

A PERTURBATION THEORETIC APPROACH
TO
THE ANALYTIC PROPERTIES OF COLLISION AMPLITUDES

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PREFACE

The author wishes to thank Professor N. Kemmer for the hospitality of the Tait Institute, and also Dr. G.R. Screation, who suggested the field of the present inquiry, for his continued interest and guidance in the work described in this dissertation. To all members of the Tait research school, and in particular to his above named supervisors, the author is indebted for many helpful and stimulating discussions during the course of his studies in Edinburgh.

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The material presented in this thesis is asserted to be original except insofar as explicit reference has been made to the work of others.

Edinburgh,
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INTRODUCTION

In recent years the quantum theory of fields has become increasingly more complicated, and, despite the remarkable successes of the theory, it is widely held that, without some new simplifying principle, there is little future for this theory as an effective tool in the study of elementary particle interactions. At the present time the best hope of building a new and powerful theory seems to lie in the development of the method of dispersion relations. The basic idea of dispersion relations is that the quantum mechanical amplitudes, which describe physical processes, are the boundary values of functions of one, or several, complex variables^{*}, regular apart from poles in a suitably cut space: an early indication of this notion was given in 1955 by Chew and Low⁽²⁾ who discovered that, in the static model for elastic scattering of a pion on a nucleon, one was dealing with an amplitude, meromorphic in the energy plane except for branch cuts lying along the real axis. Dispersion relations are based on very broad general principles such as covariance, spectral conditions, and locality. In the so-called axiomatic approach to dispersion theory one attempts to deduce from these basic principles, in a rigorous fashion, the analytic properties of the collision amplitudes as functions of complex invariants. Then one is able

* the invariants of the problem.

to exploit a knowledge of the location and nature of the singularities of an amplitude to derive a useful integral representation for it using Cauchy's integral theorem. These representations are called "dispersion relations" or "spectral representations."

The axiomatic approach is quite complicated and it is exceedingly difficult to establish dispersion relations for many processes of real physical interest. None the less, dispersion relations have been written down, on non-rigorous grounds, for a variety of interactions, and their use has already produced a startling wealth of results which have been checked successfully against experimental data: not only this, but on dispersion theoretic grounds, important experimental predictions have been made. A good example of this is the ⁽³⁾ now well-known Fraser and Fulco prediction that, in the $j = 1$, $t = 1$, state of the pion-pion system, there must occur a resonance at around 450 MeV. Fraser and Fulco analysed electron-nucleon scattering experiments, using dispersion theory, in order to discover the extent of the spatial distribution of the nucleonic charge and magnetic moment. Loosely, they pictured the electron as interacting with the nucleon's pion cloud: the nucleon emits a pion, which interacts with an electron, and is then reabsorbed (see Fig. 1): pion structure is, therefore, of fundamental importance in the investigation. The problem of pion structure can in turn be related to the

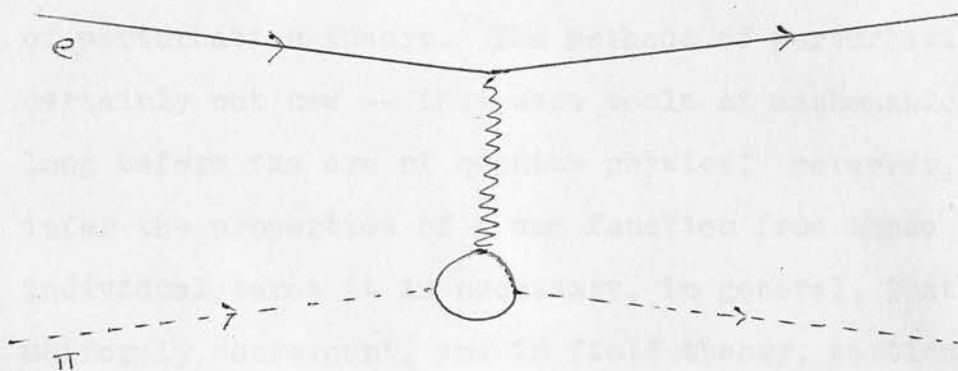
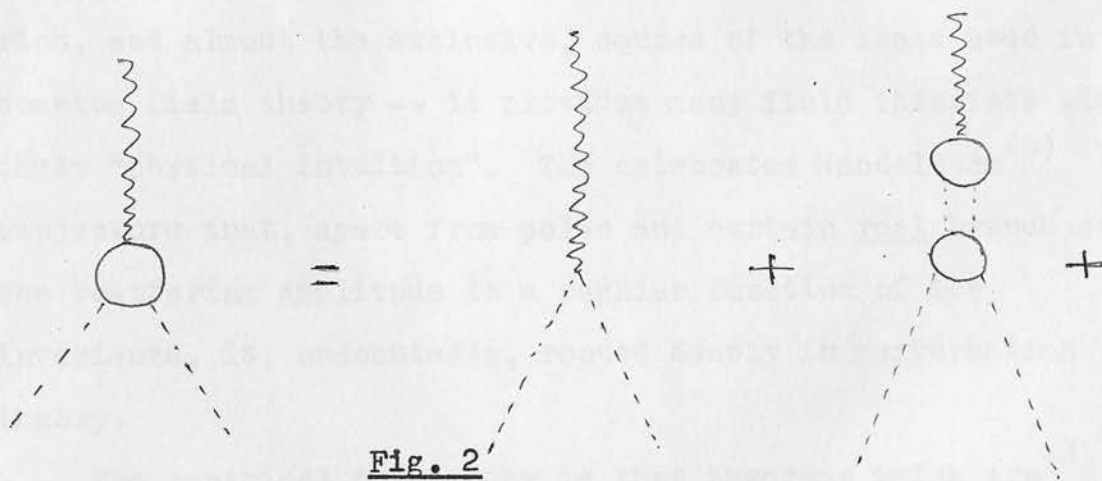


Fig. 1

(lowest order in fine structure constant)

pion-pion interaction, in the sense that the two pion exchange term contribution (see Fig. 2) may be adjusted to make the theory fit the original electron-nucleon scattering data. The actual predicted value, as calculated by Fraser and Fulco, has since been shown to be in good qualitative agreement with



experiment: their theoretical work was the first real hint of the existence of this important resonance.

The present author's work has been concerned with the analytic structure of collision amplitudes within the framework

of perturbation theory. The methods of perturbation theory are certainly not new -- they were tools of mathematical physicists long before the era of quantum physics; moreover, in order to infer the properties of a sum function from those of its individual terms it is necessary, in general, that the sum be uniformly convergent, and in field theory, particularly in the case of strong couplings, grave doubts exist regarding even simple convergence of this sum -- indeed, no one has demonstrated that the perturbation series can be summed successfully!

Why then, one may well ask, study perturbation methods?

The answer is not simply that mathematical physicists do not worry about such niceties as uniformity of convergence.

While the axiomatic approach is often too complicated to handle, perturbation theory has been, and continues to be, a rich, and almost the exclusive, source of the ideas used in quantum field theory -- it provides many field theorists with their "physical intuition". The celebrated Mandelstam⁽⁴⁾ conjecture that, apart from poles and certain real branch cuts, the scattering amplitude is a regular function of its invariants, is, undoubtedly, rooted deeply in perturbation theory.

The empirical philosophy is that theorems which are demonstrably false in some order of perturbation theory are to be viewed with scepticism, while those theorems capable of proof to all orders in perturbation theory are worthy of further consideration. This philosophy has, in practice, been

a very fruitful one -- and, quite bluntly, the fact must be faced that, in most problems, the perturbation theoretic version is the only one which can be tackled at the present stage of development of the subject.

Even so, in the face of a lack of mathematical justification, all this is simply the faith of the perturbation theorist in the naive approach: the author's conviction that perturbative sources will lead to further important advances may not convince others.

However, in applying perturbation theory to dispersion relation techniques, the author can offer the mathematical purist and the sceptic some further justification for his standpoint. In dispersion theory, the position and nature of the singularities of collision amplitudes are the core of the problem. It is widely held that the singularities of perturbation theory are controlled by unitarity (which is essentially the conservation of probability) and it has been shown by Polkinghorne⁽⁵⁾ that the set of singularities of the amplitudes predicted by perturbation theory form the minimum set which is consistent with unitarity. Thus, while it is just conceivable that the amplitudes possess singularities not given by perturbation theory, it is certain (unless there is some accidental cancellation) that all the singularities which are found in perturbation theory are relevant for any unitary field theory.

As we have already emphasised, the axiomatic approach is exceedingly laborious and very few dispersion relations

can actually be proved. On the other hand, dispersion relations written down on the evidence of perturbation theory (and to a lesser extent on the evidence supplied by potential theory in non-relativistic quantum mechanics*) have met with unprecedented success. We are forced to the stage of asking ourselves whether it is worth while continuing with the old axiomatic approach -- indeed, the status and consistency of the axioms of field theory have been questioned by some writers. In perturbation theory which can be derived non-rigorously from the axioms, we have explicit expressions for each term in the series for an amplitude, and if the method can be applied exhaustively to various interactions, one might hope to utilise the perturbation theoretic predictions to postulate the analytic structure of a new theory. The situation towards which we are moving is, in fact, a new starting point for the theory of elementary particle interactions based on a few simple new axioms, one of which would relate to analytic structure: the others would probably be concerned with unitarity and crossing symmetries. We would then be in a position to discard the old axioms and start afresh. Such a scheme has been proposed by Chew.⁽¹⁾

This, then, is the background against which the author has undertaken to write a thesis entitled "A perturbation theoretic approach to the analytic properties of collision amplitudes." The dissertation is a chronological account of the author's studies of perturbative methods, and falls, broadly speaking,

* At the time of writing the technique of employing complex angular momenta in non-relativistic quantum mechanics has proved of great value, and calculations on these lines have become very fashionable.

into two distinct parts. The first part, which serves as an introduction to the subject, describes properties of the Landau curves which, as will be discussed later, are loci, in the multidimensional complex space of the invariants, corresponding to points of possible singularity of a collision amplitude. At the time of commencing the present work (Summer 1960) the author had participated in the Scottish Universities' Summer School which was devoted to the subject of dispersion relations: also several papers, in preprint form, dealing with the elastic scattering of two particles in perturbation theory, had recently come to hand. These two circumstances inevitably coloured the author's entry into the subject, and some of his earlier work consisted in the elaboration of points raised in the literature, or else arose, directly or indirectly, from discussions of then current topics with Dr. G.R. Screaton, the author's research supervisor, himself actively engaged in work on perturbation theory.

The second section gives a detailed account of an independent investigation carried out by the author into the problem of finding an integral representation for production processes in perturbation theory. The type of process considered is characterized by the inelastic scattering of a pion on a nucleon:



The outcome of this investigation was quite surprising

for it brought to light a new problem hitherto unencountered in simple elastic scattering processes,* namely the occurrence of complex branch points which prevent the writing of conventional single dispersion relations for any physical values of the fixed invariants. One is thus faced with the problem of fitting this new feature into the existing scheme, and, in general, the matter appears to be an exceedingly complicated one: the urgency of coping with complex branch cuts becomes even more intense because of the recent discovery by Polkinghorne that complex singularities can arise as a direct consequence of the unitarity assumption. Unitarity is one of the basic physical principles of the theory.

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* It is now known that, in the perturbation series for the elastic scattering amplitude, there do occur terms which possess complex singularities for certain ranges of mass values.

** See also G.R. Screaton (Ed.) "Dispersion Relations" published Oliver & Boyd, 1960.

CHAPTER 1

SOME PROPERTIES OF THE LANDAU CURVES

(1.1) The Landau Equations

The terms of the perturbation series for a given collision amplitude are of the form

$$F(z) = \lim_{\varepsilon \rightarrow 0+} \int d^4k_1 \cdots d^4k_n \prod_{i=1}^n (q_i^2 - m_i^2 + i\varepsilon)^{-1} \quad (1.1.1)$$

where q_i is the four-momentum of the particle of mass m_i , which corresponds to the i -th internal line of a Feynman diagram: q_i depends linearly, via the law of conservation of energy momentum at each vertex, on a set of independent internal momenta k_i and on the external momenta p_i . The symbol z summarises a total set of independent scalar variables z_i which can be constructed in an invariant manner from the vectors p_i .

The complications of charge and spin dependence have been ignored on the grounds that the factors which these considerations introduce, occurring as they do in the numerator of the integrand, cannot increase the number of singularities of the function $F(z)$. Of course, this means that we have discarded important selection rules which we will require to impose artificially when necessary. However, as the perturbation approach is intended as a model for a more sophisticated theory, simplicity is of primary importance.

By means of a transformation due to Feynman we obtain:

$$F(z) = \lim_{\varepsilon \rightarrow 0+} \int d^4k_1 \dots d^4k_n \int_0^1 \frac{d\alpha_1 \dots d\alpha_n \delta(1 - \sum_{i=1}^n \alpha_i)}{d(q, \alpha, \varepsilon)^n} \quad (1.1.2)$$

where

$$d(q, \alpha, \varepsilon) = \sum_{i=1}^n \alpha_i (q_i^2 - m_i^2 + i\varepsilon) \quad *$$

In dispersion theory, we wish to locate the singularities of the functions $F(z)$. The sets of values of the variables z_i , both real and complex, which correspond to possible singularities of $F(z)$ are called the Landau curves, and may be obtained by processes of elimination^{**} from a set of algebraic equations first written down by Landau⁽⁴⁾.

The Landau equations for an uncontracted diagram are:

$$\sum_{i=1}^n \alpha_i = 1 \quad (1.1.3)$$

$$q_i^2 = m_i^2 \quad (1.1.4)$$

$$\sum_i \alpha_i q_i = 0 \quad (1.1.5)$$

together with equations which express the law of conservation of four-momentum at each vertex. There exists an equation of the type

* The symbols q and α summarise respectively the variables q_i and α_i .

** In the appendix, (equations (1) - (10)), the author will show, by an example, how such an elimination may be carried out. The author's method leads not only to the usual Landau singularities but also to the recently discovered non-Landau singularities.

(1.1.4) corresponding to each internal line of the diagram: there are ℓ equations of the type (1.1.5), where the summations are taken round independent closed loops of the diagram.

Equivalent criteria have also been given by Polkinghorne and Screatton⁽⁸⁾. They proceed by first performing the k integrations to obtain

$$F(z) = \lim_{\varepsilon \rightarrow 0+} \int_0^1 \frac{d\alpha_1 \dots d\alpha_n \delta(1 - \sum_{i=1}^n \alpha_i) f(\alpha, n, \ell)}{D(z, \alpha, \varepsilon)^{n-2\ell}} \quad (1.1.6)$$

Their philosophy is to consider the multiple integral as an integral over a "contour" A in α -space (i.e. in the space of the variables α_i): at any point z there will exist a set of points $\alpha(z)$ at which $D(z, \alpha, \varepsilon)$ vanishes, and as z is varied the analytic continuation of $F(z)$ will be obtained by deforming A to avoid the zeros of D : in this way we obtain the singularities of $F(z)$ ^{*} when such deformations become impossible, i.e. when either two zeros of D pinch the contour between them (Fig. 3) or a zero of D moves up to the fixed boundary of the contour (Fig. 4). The equations of Polkinghorne and Screatton which express these two alternatives are respectively:

$$\left. \begin{aligned} \frac{\partial D}{\partial \alpha_{v_i}} &= 0 & i &= 1, \dots, t \\ \alpha_{v_i} &= 0 & i &= t+1, \dots, n \end{aligned} \right\} \quad (1.1.7)$$

* In general $F(z)$ is a many valued function and the analytic behaviour at any point will depend on the mode of continuation.

for any partition of the α_{ν_i} into classes of r and $n - r$ members, $0 < r \leq n$. When $r = n$ we talk of the resulting Landau

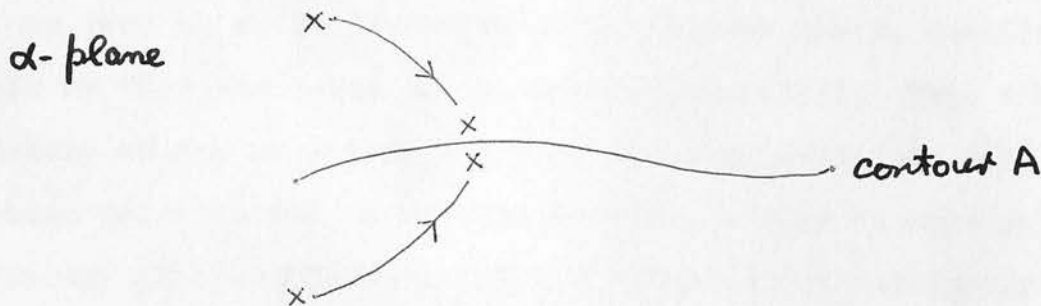


Fig. 3*

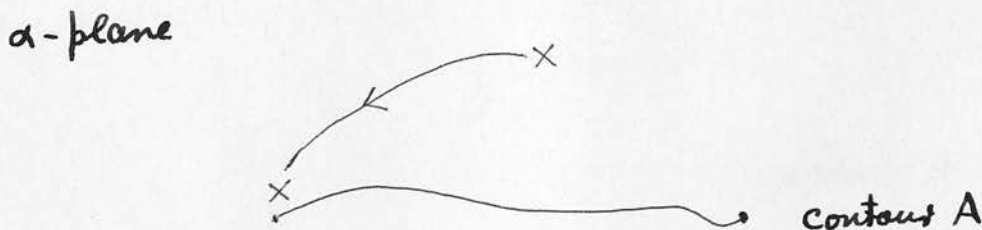


Fig. 4.

curve as the leading curve, and when $r < n$ of a lower order curve. Each partition of the α_{ν_i} in (1.1.7) corresponds to a complete set of Landau equations with non-zero α_{ν_i} . The leading curve gives the singularities belonging to the uncontracted Feynman graph while the lower-order curves correspond to the contracted diagrams.

* These figures are to be regarded as 2-dimensional models of the $2n$ -dimensional situation. The full justification for drawing such pictures involves performing $n - 1$ of the integrations in (1.1.6).

(1.2) Effective Intersections

Suppose that we are at a point of singularity of a function $F(z)$ which corresponds to a pinching of the contour as in Fig. 3. If we vary z , while remaining on the Landau curve, equations (1.1.7) tell us that the zeros of D remain coincident. Thus a mechanism whereby we may move from a region of singularity of $F(z)$ on a Landau curve to one of non-singularity is that we move up to a boundary of the contour A and the pinch occurs harmlessly thereafter as in Fig. 5. This mechanism, for a long time supposed to be

α -plane



Fig. 5.

the only mechanism for moving from regions of singularity to ones of non-singularity, is very important. At the point of transition we reach the boundary of the contour and hence the Landau curve on which we were varying our z -value has intersected a lower order curve. We define an intersection between a Landau curve Σ and a lower order curve Σ' , at which the Landau equations for each are satisfied by the same set of α_i values, to be an effective intersection. At effective intersections the analytic properties of $F(z)$ may change.

Tarski⁽⁹⁾ has proved, in the special case of single loop diagrams, that a Landau curve Σ intersecting a curve Σ' of next lowest order necessarily does so effectively.

We now proceed to discuss the validity of the more general assertion that this theorem applies to Landau curves belonging to any arbitrary Feynman diagram.

Consider the set of equations

$$\frac{\partial D}{\partial \alpha_i} = 0 \quad i=1, \dots, n-1. \quad (1.2.1)$$

which, as follows immediately from equations (1.1.2) and (1.1.6) are homogeneous of degree zero in the variables α_i .

Suppose, for the present, that we may solve equations (1.2.1), uniquely, in the following fashion:

$$\alpha_1 : \alpha_2 : \dots : \alpha_n = A_1 : A_2 : \dots : A_n \quad (1.2.2)$$

where the A_i are algebraic functions of the variables z_i , from which we have removed any infinities.

The equation of the curve Σ' may be constructed in the following manner. In addition to (1.2.2) impose the further condition that $\alpha_n = 0$: it follows at once that the equation of Σ' is simply given by:

$$A_n(z) = 0 \quad (1.2.3)$$

To construct the equation of the curve Σ the condition which must be imposed in addition to (1.2.2) is $\frac{\partial D}{\partial \alpha_n} = 0$. If we now write

$\frac{\partial D}{\partial a_n} = E$ then E is homogeneous of degree zero in the a_i variables. It follows that E can be expressed as a function of the ratios of the a_i 's only. Hence the equation of Σ is:

$$E(A_1, A_2, \dots, A_n) = 0 \quad (1.2.4)$$

The theorem now follows: our construction ensures that Σ and Σ' have the same set of a_i values at their points of intersection.*

However, it may turn out that the solution (1.2.2) is not unique. In this case the theorem may fail to hold when, corresponding to a given value of z , there exists more than one set of functions $A_i(z)$. If there happens to be an N -fold degeneracy we must rewrite equations (1.2.2) in the form

$$\alpha_1 : \alpha_2 : \dots : \alpha_n = A_1^i : A_2^i : \dots : A_n^i \quad (1.2.2A)$$

where

$$i = 1, 2, \dots, N.$$

Now the equations of the curves Σ' and Σ will be respectively

$$\bigcap_{i=1}^N A_n^i(z) = 0 \quad (1.2.3A)$$

and

$$\bigcap_{i=1}^N E^i(z) = 0 \quad (1.2.4A)$$

* It should be emphasised that the argument does not hold good when more than one of the a_i vanishes and consequently the proof relates only to intersections of curves whose orders differ by unity.

where

$$E^i(z) = E(A_1^i, A_2^i, \dots, A_n^i)$$

There is no longer any guarantee that the intersection of $A_n^i(z) = 0$ with $E^j = 0$ will be effective when $i \neq j$, although certainly the intersections of these curves will be effective if $i = j$.

The author's conclusion is thus that in the simplest conceivable case, namely when to each point z of a Landau curve there corresponds a unique set of α_i values, curves of consecutive order intersect effectively regardless of the complication of the Feynman diagram. When the α_i 's corresponding to a given z -value are degenerate the intersection may or may not be effective: we can give no example of Landau curves of consecutive orders intersecting non-effectively but we can see no reason why such behaviour should not occur for some diagram: further consequences of degenerate behaviour are discussed in Section (1.3).

Landshoff, Polkinghorne and Taylor⁽⁵⁾ have shown, in the case of two invariants, that at effective intersections of Landau curves, the curves have parallel tangents.

In the general case the proof of the tangency property is still very simple. It can be shown that the denominator function $D(z, \alpha)$ in equation (1.1.6) can be written as

$$D(z, \alpha) = \sum_{i=1}^n b_i(\alpha) z_i + b_{n+1}(\alpha) \quad (1.2.5)$$

Now the direction of the normal to the tangent hyper-plane of the Landau curve will be given by the set of ratios:

$$\frac{\partial D}{\partial z_1} : \frac{\partial D}{\partial z_2} : \dots : \frac{\partial D}{\partial z_n} \quad (1.2.6)$$

Differentiating equation (1.2.5) with respect to z_i we obtain

$$\frac{\partial D}{\partial z_i} = f_i(\alpha) + \sum_j \frac{\partial D}{\partial \alpha_j} \frac{\partial \alpha_j}{\partial z_i} = f_i(\alpha) \quad (1.2.7)$$

on both Σ and Σ' , using the Landau equations in the form (1.1.7). Thus, at an effective intersection between Σ and Σ' , the ratios (1.2.6) are identical and the curves touch.

These properties provide a simple geometric way of classifying the Landau curves into families which touch one another in prescribed fashions. Such a classification was given by Tarski in the case of the single loop scattering diagram. A similar classification for curves corresponding to a five-point single loop graph was attempted by the present author, and has also been given recently by Cook and Tarski⁽¹⁾. This latter classification has not, as yet, proved to be of any value whatsoever and so we will not discuss it here. In the former case, however, Tarski proceeded to a satisfying and elegant verification of the truth of the Mandelstam representation, for certain external mass values, for the four-point single loop graph. Tarski's proof of double dispersion relations for the simplest scattering diagram forms the basis of the more general discussions of Landshoff, Polkinghorne

and Taylor. Their methods are inductive, and properties of the scattering diagram to any order in perturbation theory are asserted -- the single loop graph being the starting point of the induction procedure.

An essential feature of the proofs of the Mandelstam representation is that the effective intersections can, in some sense, divide up the Landau curves into regions, each of which corresponds to a definite type of analytic structure for $F(z)$.^{*} One may well ask in what manner this is possible because the set of effective intersections is of too low a dimensionality to divide up the Landau curves! In general $F(z)$ is a many valued function and our interest is centred upon one specific sheet, namely the physical sheet. Thus, if we perform analytic continuations of $F(z)$ by paths lying on the Landau curves, we must avoid passing through branch cuts on to unphysical sheets, or if we do enter such sheets we must ensure that we return eventually to the correct sheet. It happens that we can sometimes do neither of these things without being forced to pass through an effective intersection -- and an effective intersection is a point through which continuations may not be made. A detailed analysis of the mechanisms involved is given in Section (1.4).

(1.3) Modes of Continuation

Because of the reality of the Landau equations the Landau curves are real in the sense that they are algebraic curves with

* We ignore, for the present, those points other than effective intersections at which the analytic properties of $F(z)$ may change.

real coefficients. As a result, a Landau curve can, in some measure, be represented by its real section, i.e. that portion of the curve corresponding to choosing all the invariants real. The complex regions of the curve can then be obtained by using a generalisation of the 'search-line' technique introduced by Tarski for the case of two invariants. For a Landau curve depending on two invariants only, this is to say that all points of a Landau curve are found by taking its real section together with the complex intersections of the curve with the set of all possible real lines. In the general case we must take the real section together with the curve's complex intersections with all possible real hyper-planes.* In Fig. 6, for example, this technique tells us that a whole double region of the curve composed of complex points, joins the two arcs: a family of search-lines of constant gradient maps out points which sprout off each arc in complex conjugate pairs; the entire complex region of the curve is then obtained by varying the gradient of the search line.

Evidently enough, the technique is not very informative in the case of Landau curves with no real section (e.g. $x^2 + y^2 + 1 = 0$ is composed entirely of complex points). If such curves do exist, then, unless they are composed entirely of points corresponding to regular behaviour, it may be impossible even to write a simple single dispersion relation in any invariant.

The basic method of determining whether or not a function $F(z)$

* By a real hyper-plane we mean, in the case of n invariants, any manifold of the form

$$\sum \lambda_i z_i = c$$

where λ_i and c are real.

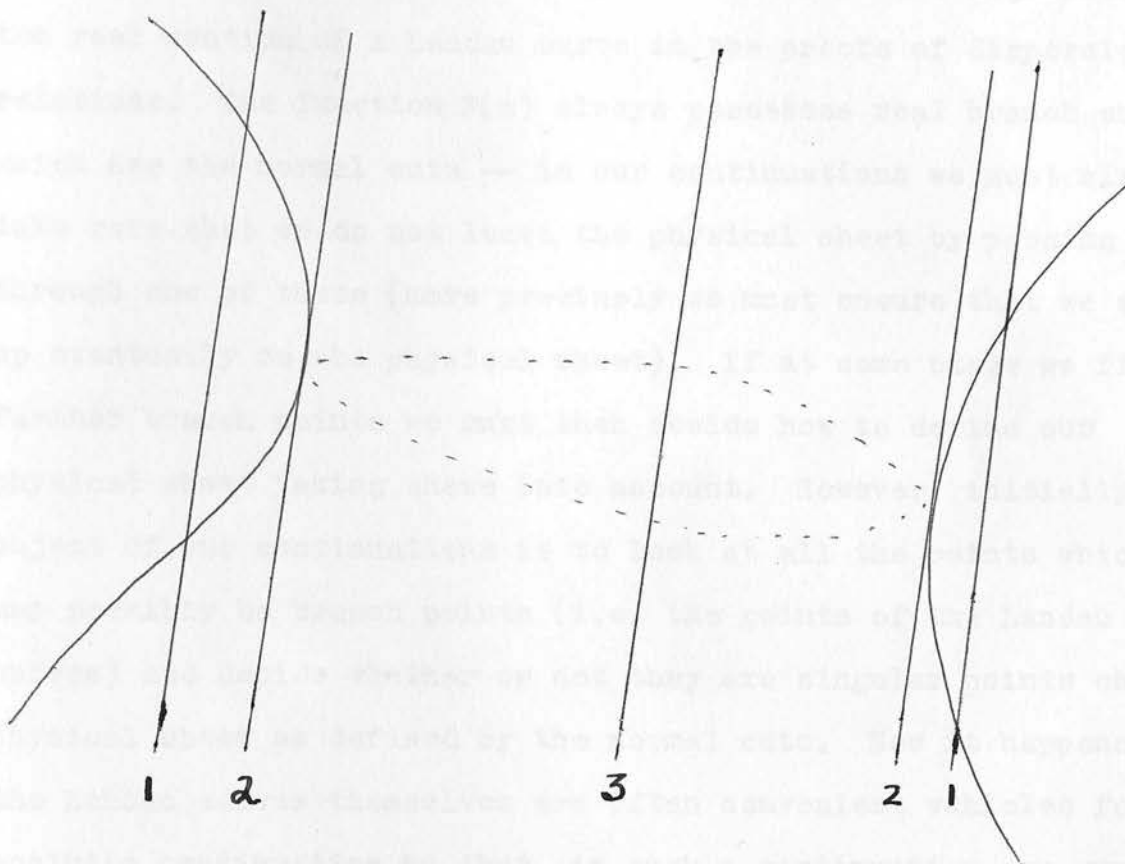


Fig. 6

- 1 = Real intersections
- 2 = Coincident intersections
- 3 = Complex intersections

is singular is to continue the function from a region of the z -plane where it is mathematically well defined to the region of interest, taking into account that a singularity may arise whenever we encounter a Landau curve. In general the $F(z)$ will be many valued and the mode of continuation will affect the singularity or non-singularity at a point.*

* In the appendix a concrete example is studied in detail.

This last paragraph gives the clue to the true significance of the real section of a Landau curve in the proofs of dispersion relations. The function $F(z)$ always possesses real branch cuts which are the normal cuts -- in our continuations we must always take care that we do not leave the physical sheet by passing through one of these (more precisely we must ensure that we end up eventually on the physical sheet). If at some stage we find further branch points we must then decide how to define our physical sheet taking these into account. However, initially the object of our continuations is to look at all the points which may possibly be branch points (i.e. the points of the Landau curves) and decide whether or not they are singular points on the physical sheet as defined by the normal cuts. Now it happens that the Landau curves themselves are often convenient vehicles for analytic continuation so that, in such a continuation, we must exercise great care on the real section, because on moving on it we may pass through a normal cut into an unphysical sheet. Thus it is that the real section plays a central rôle in this theory.

If we continue the function $F(z)$ from a region of the real z -plane where $F(z)$ is regular, to a point z_1 , then the continuation by the complex conjugate route to z_1^* yields the same type of analytic behaviour for $F(z)$ at both z_1 and z_1^* . This, as Landshoff, Polkinghorne and Taylor remark, is because the complex conjugate mode of continuation in the z -plane leads us, in the α -plane, to the complex conjugate configuration of both

the contour A and the zeros $a(z)$ of D .

Let us suppose that we have a real point z_0 at which $F(z)$ is regular, and let us choose a path from z_0 to z_1 which does not intersect any portion of the Landau curve Σ (that it is indeed possible to construct such a path is to be proved later in this section).

d -plane

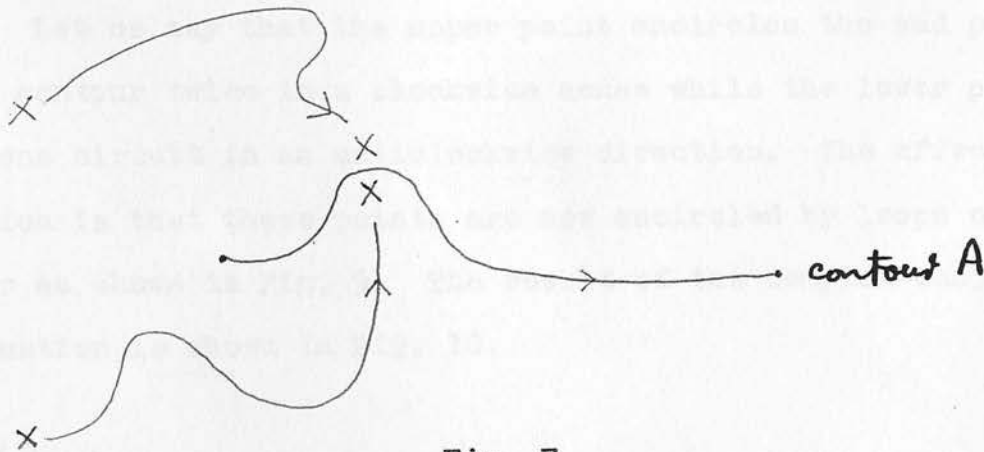


Fig. 7.

d -plane

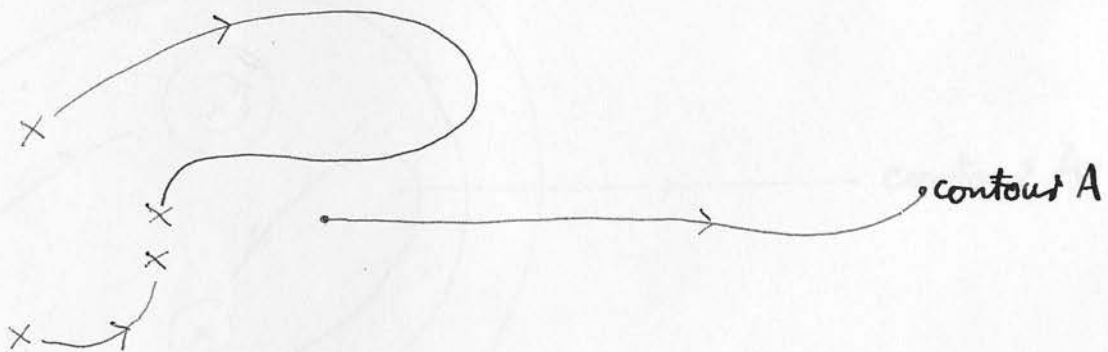
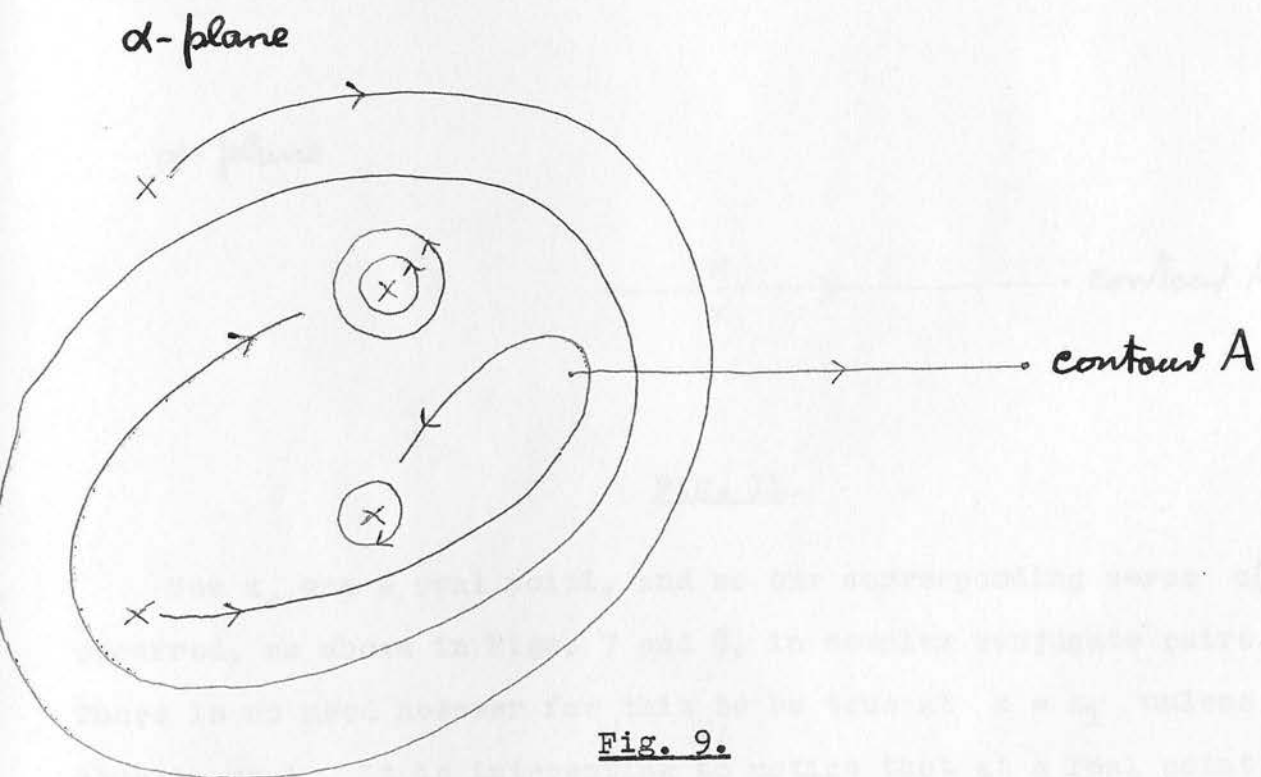


Fig. 8.

Fig. 7 depicts a pair of zeros $a(z)$ moving, as z varies from z_0 to z_1 , into a coincidence which pinches the contour and

produces a singularity of $F(z)$ at $z = z_1$. On the other hand, in Fig. 8, we have a coincidence which corresponds to regular behaviour at $z = z_1$. It is fairly evident that the complex conjugate configurations lead to identical behaviour at $z = z_1^*$. More complicated situations are clearly conceivable because the zeros of D may move in such a way as to necessitate drastic contour deformation. As a further example, consider a pair of points which move from the initial to final configurations of Fig. 8 but via more tortuous paths. Let us say that the upper point encircles the end point of the contour twice in a clockwise sense while the lower point makes one circuit in an anticlockwise direction. The effective situation is that these points are now encircled by loops of the contour as shown in Fig. 9. The result of the complex conjugate continuation is shown in Fig. 10.



α -plane

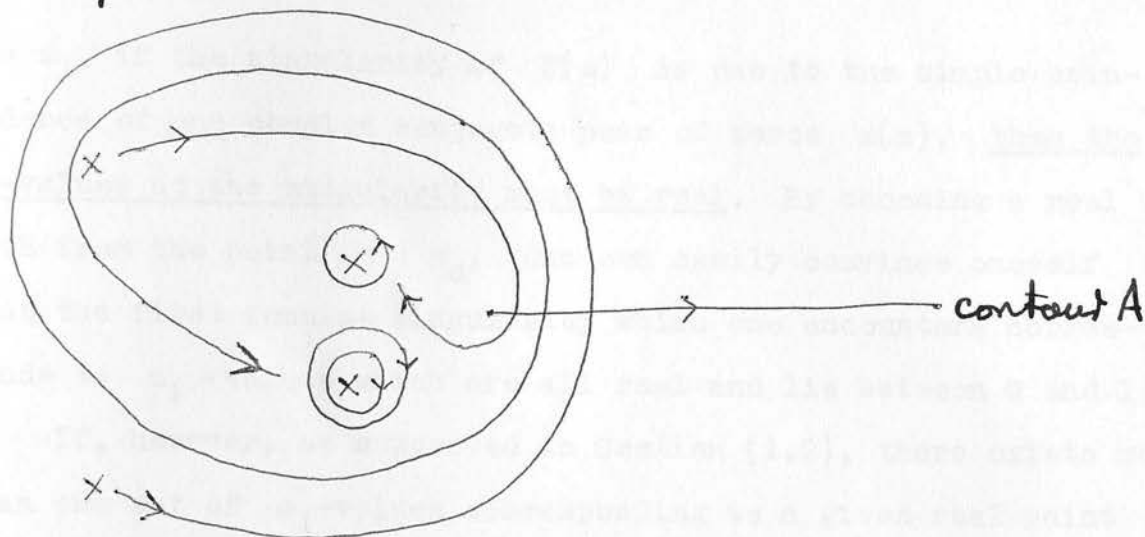


Fig. 10.

Both Fig. 9 and Fig. 10 depict pinches of the type illustrated in Fig. 11, and so both modes of continuation lead to identical types of singularity for $F(z)$.

α -plane



Fig. 11.

Now z_0 was a real point, and so our corresponding zeros $\alpha(z_0)$ occurred, as shown in Figs. 7 and 8, in complex conjugate pairs. There is no need however for this to be true at $z = z_1$ unless z_1 also is real. It is interesting to notice that at a real point

$z = z_1$, if the singularity of $F(z)$ is due to the simple coincidence of one complex conjugate pair of zeros $\alpha(z)$, then the α_i -values at the singularity must be real. By choosing a real path from the point $z = z_0$, one can easily convince oneself that the first genuine singularity which one encounters corresponds to α_i -values which are all real and lie between 0 and 1.

If, however, as suggested in Section (1.2), there exists more than one set of α_i -values corresponding to a given real point $z = z_1$ of a Landau curve then it is possible for the α_i 's to be complex. Fig. 12 illustrates such a situation. It is not the

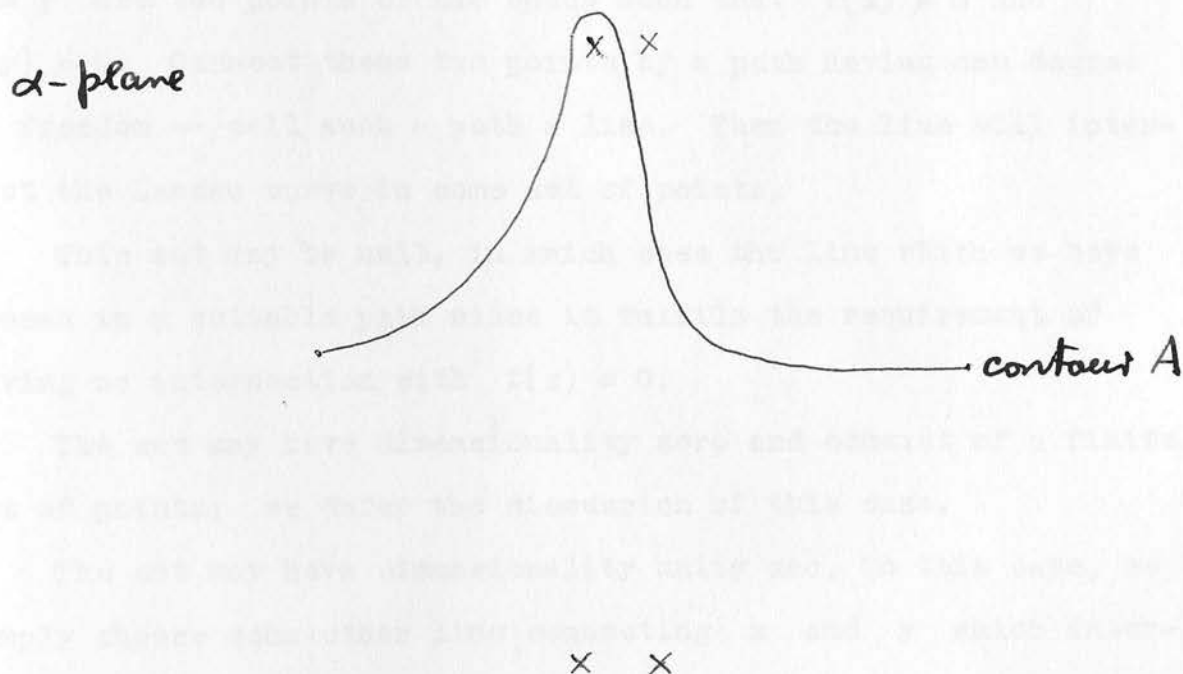


Fig. 12.

coincidence of a complex conjugate pair of points $\alpha(z)$ which gives rise to the pinching of the contour, but a coincidence between two such pairs. It is clear that the complex conjugate configuration will also produce a singularity.

All possible situations lead to the same conclusions.

We now proceed to prove that we can, in fact, continue in the complex z -plane by paths which never intersect a Landau curve.

Let the equation of the Landau curve be $f(z) = 0$ where z summarises n variables z_1, z_2, \dots, z_n . Essentially there are $2n$ real variables and $f(z) = 0$ gives two real equations. Thus the Landau curve is a manifold of dimensionality $2n - 2$ in a space whose dimensionality is $2n$. Suppose $z = x$ and $z = y$ are two points of the space such that $f(x) \neq 0$ and $f(y) \neq 0$. Connect these two points by a path having one degree of freedom -- call such a path a line. Then the line will intersect the Landau curve in some set of points.

This set may be null, in which case the line which we have chosen is a suitable path since it fulfils the requirement of having no intersection with $f(z) = 0$.

The set may have dimensionality zero and consist of a finite set of points: we defer the discussion of this case.

The set may have dimensionality unity and, in this case, we simply choose some other line connecting x and y which intersects the Landau curve in a set of points falling into either the first or the second category. It is always possible to do this because if it were not possible the manifold $f(z) = 0$ would have dimensionality greater than $2n - 2$.

Thus our problem is essentially whether or not we can find our way past a single point C on a Landau curve: we assert that it is a trivial matter to modify our path through C only slightly to avoid C and obtain a line which does not intersect the Landau curve at all. Suppose that the coordinates of the point C are z_c . In the neighbourhood of $z = z_c$ the Landau curve has the form

$$f(z_c + \{ \}) = f(z_c) + \left. \frac{\partial f}{\partial z} \right|_{z=z_c} \cdot \{ \} = \left. \frac{\partial f}{\partial z} \right|_{z=z_c} \cdot \{ \} = 0 \quad (1.3.1)$$

provided only that $\left. \frac{\partial f}{\partial z} \right|_{z=z_c} \neq 0$ for some of the variables z_c .

We can write the equation of the Landau curve locally as

$$\sum_i a_i \{ \}_i = 0 \quad (1.3.2)$$

where

$$a_i = \left. \frac{\partial f(z)}{\partial z_i} \right|_{z=z_c} \quad (1.3.3)$$

Equation (1.3.2) is, in general, two real equations which we can write as follows:

$$\sum_i \text{Re} a_i \text{Re} \{ \}_i - \sum_i \text{Im} a_i \text{Im} \{ \}_i = 0 \quad (1.3.4)$$

$$\sum_i \text{Im} a_i \text{Re} \{ \}_i + \sum_i \text{Re} a_i \text{Im} \{ \}_i = 0 \quad (1.3.5)$$

The condition that (1.3.4) and (1.3.5) should be the same equation is

$$\frac{\operatorname{Re} a_i}{\operatorname{Im} a_i} = - \frac{\operatorname{Im} a_i}{\operatorname{Re} a_i} \quad \text{for all } i=1,2,\dots,n, \quad (1.3.6)$$

which is simply an expression of the condition

$$|a_i| = \left| \left(\frac{\partial f}{\partial z_i} \right) \Big|_{z=z_c} \right| = 0 \quad (1.3.7)$$

which we have already assumed untrue in writing equation (1.3.1).

Now (1.3.4) and (1.3.5) define a subspace of dimensionality $2n - 2$. Let us take a set of basis vectors to span our original $2n$ -dimensional space, the first $2n - 2$ of which actually span the subspace defined by equations (1.3.4) and (1.3.5).

Then

$$\left. \begin{aligned} x &= (x_1, x_2, \dots, x_{2n-2}, x_{2n-1}, x_{2n}) \\ y &= (y_1, y_2, \dots, y_{2n-2}, y_{2n-1}, y_{2n}) \end{aligned} \right\} \quad (1.3.8)$$

where not both of the last two coordinates may vanish since x and y do not lie on the Landau curve. We now construct a path from x to y which does not intersect $f(z) = 0$. Several cases must be distinguished. In all cases we first move

$(x_1, \dots, x_{2n-2}) \longrightarrow (y_1, \dots, y_{2n-2})$ in a continuous manner.

Case 1.

$$x_{2n-1}, x_{2n}, y_{2n-1}, y_{2n} \neq 0$$

Let $x_{2n-1} \rightarrow y_{2n-1}$, and then $x_{2n} \rightarrow y_{2n}$: or make the variations in the opposite order. In either case it is impossible

for both coordinates to vanish simultaneously.

Case 2

$$\left. \begin{aligned} x_{2n-1} &= 0, & x_{2n}, y_{2n-1}, y_{2n} &\neq 0 \\ y_{2n} &= 0, & x_{2n-1}, x_{2n}, y_{2n-1} &\neq 0 \\ x_{2n-1} &= y_{2n} = 0, & x_{2n}, y_{2n-1} &\neq 0 \end{aligned} \right\}$$

Let $x_{2n-1} \rightarrow y_{2n-1}$ and then $x_{2n} \rightarrow y_{2n}$, the order of the operations being essential in this case to avoid the possibility of both coordinates being zero at once.

Case 3

$$x_{2n-1} = y_{2n-1} = 0, \quad x_{2n}, y_{2n} \neq 0$$

Let $x_{2n-1} \rightarrow \varepsilon \neq 0$, then $x_{2n} \rightarrow y_{2n}$, and

finally let $\varepsilon \rightarrow 0$. In this way both coordinates can never vanish at the same time.

Thus a path can always be found in the neighbourhood of C which connects x and y and which does not intersect $f(z) = 0$.

If $\frac{\partial f}{\partial t_i} = 0$ for all i at the point C then $f(z) = 0$ has a multiple point at $z = z_c$. Unless all neighbouring points are also multiple points we can choose an adjacent path and use our previous argument: if, however, all neighbouring points are multiple points then $f(z)$ must have the form

$$\begin{aligned} f(z) &= g(z)^2 h(z) \\ \frac{\partial f(z)}{\partial z} &= 2g(z)h(z) \frac{\partial g(z)}{\partial z} + g(z)^2 \frac{\partial h(z)}{\partial z} \end{aligned} \quad (1.3.9)$$

and clearly it is the points of the manifold $g(z) = 0$ which are

apparently blocking our route: intersections with $g(z) = 0$ can be avoided by using the above argument again for $g(z)$ instead of $f(z)$.*

The importance of these results is to facilitate the understanding of the problem of proving the Mandelstam representation to all orders in perturbation theory. A necessary condition for the truth of this representation is the absence, on the physical sheet of our function, of complex singularities. The type of proof which is necessary, as stated in Section (1.2), is an inductive one where at any stage of the induction procedure it is assumed that all lower curves have their complex regions non-singular on the physical sheet. Thus the problem is to continue the function analytically from a region where it is known to be regular to all points of the leading curve of the diagram being considered: if we establish that a given portion of a Landau curve is non-singular it is convenient to choose our path of continuation on the curve thereafter: we must always take into account the points at which the analytic behaviour of $F(z)$ may change** -- we may not continue through these points -- and also we must notice carefully whether or not we have passed through a cut into an unphysical sheet.

* For a long time it has been supposed that Landau curves did not possess multiple points, but, recently, Eden, Landshoff, Polkinghorne and Taylor(3) have given an example of a Landau curve which does possess crunodes, and incidentally also acnodes.

** An important subset of these points are the effective intersections (see Section (1.2)) with lower order curves. Because of the induction hypothesis these all occur at real points.

(1.4) Properties of $F(z)$ on Unphysical Sheets.

Suppose that we have a function $F(z)$ of several complex variables z_1, \dots, z_n . By the results of the previous section we know that there exist, at most, 2^{n-1} distinct types of analytic behaviour of $F(z)$ as we approach a point of the real plane. For example, if $n = 3$, the limits with imaginary parts of z_i having the sign schemes $(+,+,+)$, $(+,+,-)$, $(+,-,+)$, and $(-,+,+)$ are the only possible distinct ones (those obtained by complex conjugation give identical analytic behaviour).

We now prove the following theorem: if the function $F(z)$ is singular in only one of the 2^{n-1} possible senses, then in any adjacent sheet, the function is singular in only one sense -- which sense depending upon which adjacent sheet has been chosen. For definiteness, we shall consider the case $n = 3$ with $F(z)$ singular only in the $(+,+,+)$ limit and we shall show that, in the sheets obtained by passing through the z_3 cut, $F(z)$ is singular only in the $(+,+,-)$ limit.

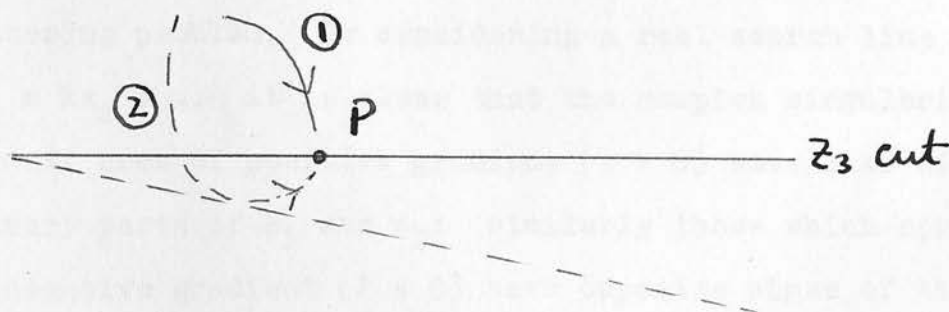


Fig. 13.

Let P , in Fig. 13, be the point under consideration. As we approach P by path (1), while the imaginary parts of z_1 and z_2 are fixed at positive values, we find a singularity at P . Let us now roll back the z_3 cut, as shown in Fig. 13, and approach P by the path (2). It is evident, then, that, in the adjacent sheet obtained by going down through the z_3 cut, $F(z)$ is singular in the $(+,+,-)$ sense. If we had fixed the imaginary parts of z_1 and z_2 at negative values and approached the z_3 cut from below the singularity would also appear because the $(+,+,+)$ and $(-,-,-)$ limits are not distinct. Thus, by rolling the z_3 cut up, we would find that $F(z)$ was singular, in the adjacent sheet obtained by going up through the z_3 cut, in the sense $(-,-,+)$. Then since $(-,-,+)$ and $(+,+,-)$ are not distinct we can assert that in both adjacent sheets $F(z)$ is singular in the $(+,+,-)$ sense. To complete the proof of the theorem we require that, in these adjacent sheets, $F(z)$ is non-singular in the senses $(+,+,+)$, $(+,-,+)$ and $(-,+,+)$. This is a trivial matter of fixing the imaginary parts of z_1 and z_2 at suitable values and again rolling back the z_3 cut.

In the particular case of $n = 2$ we are dealing with the elastic scattering problem. By considering a real search line of the form $z_1 = \lambda z_2 + \mu$, it is clear that the complex singularities which sprout off arcs of positive gradient ($\lambda > 0$) have like signs of the imaginary parts of z_1 and z_2 : similarly those which sprout off arcs of negative gradient ($\lambda < 0$) have opposite signs of the imaginary parts. One defines that limit on to the real section of a Landau curve in which the imaginary parts of z_1 and z_2 have the


* We must also roll back any other cuts which might lie in the way of our path -- this is what we mean by "adjacent".

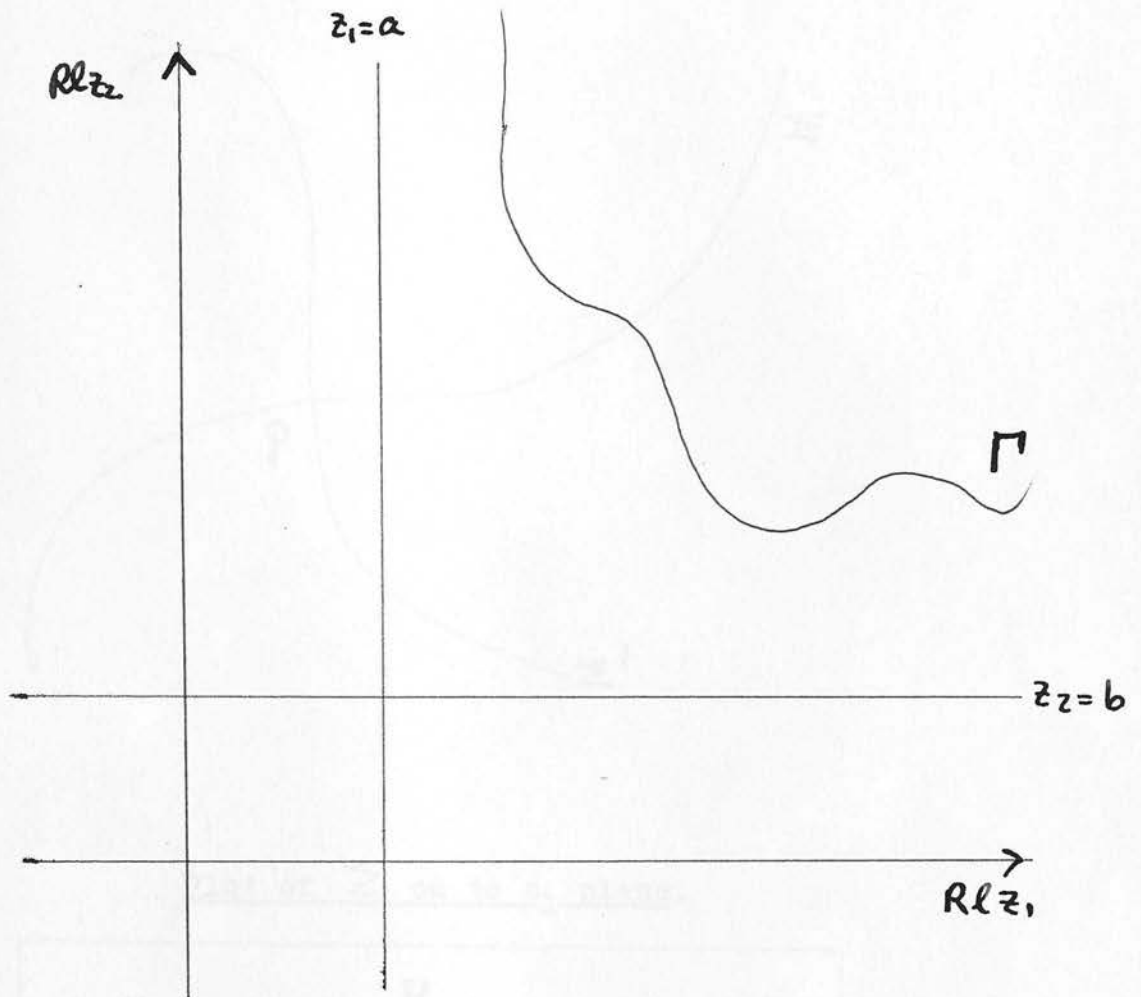
same relative sign of the imaginary parts of the Landau curve in that neighbourhood to be the appropriate limit. That limit which does not satisfy this criterion is called the inappropriate limit. Clearly the above theorem implies that a function singular only in the appropriate (or inappropriate) sense in some sheet is singular only in the inappropriate (or appropriate) sense in an adjacent sheet. Thus, a curve corresponding to an arc singular only in one sense must lie wholly inside a region which corresponds to cuts of the function in both variables: this is because, in a region below the beginning of a cut, it is immaterial whether we approach the real axis from above or below, and so appropriate and inappropriate behaviour must be identical.

In the general case of n invariants there will exist several inappropriate limits inside regions which are suitably cut.

At this stage we are now equipped to study the question raised in Section (1.2) of how a set of points, such as the effective intersections, can divide up a Landau curve into regions, each of which corresponds to identical analytic behaviour of $F(z)$. We shall consider specifically the case $n = 2$.

Let us define a plot of the Landau curve under consideration on to the z_1 plane as follows: the curve, being 2-dimensional, is capable of being projected on to the z_1 plane: a portion AB (see Fig. 14) of the real z_1 axis will correspond to the real section of the Landau curve, while other points of the plane correspond to complex points of the Landau curve.

In the example which is illustrated in Fig. 14, the real section  of the Landau curve lies wholly inside the region



Plot of Σ on to z_1 plane

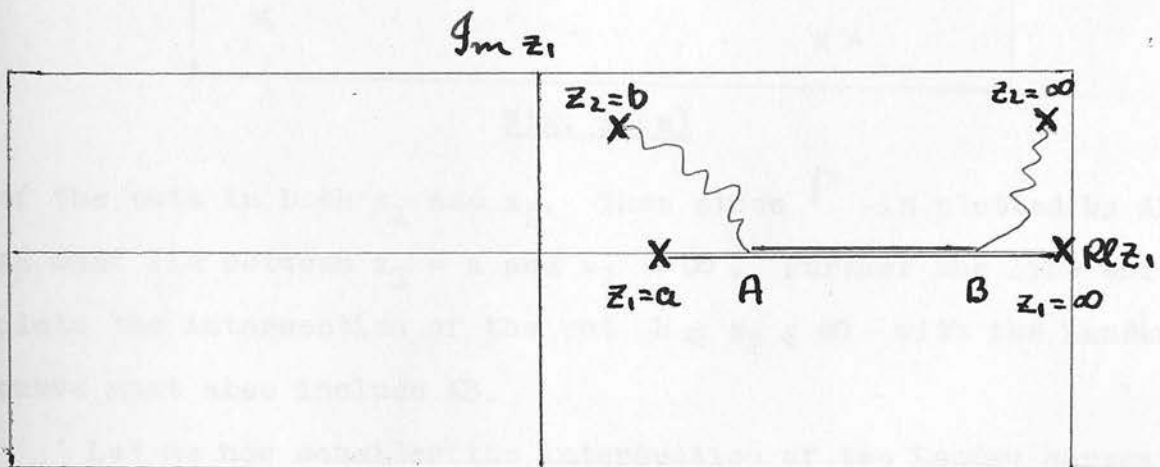
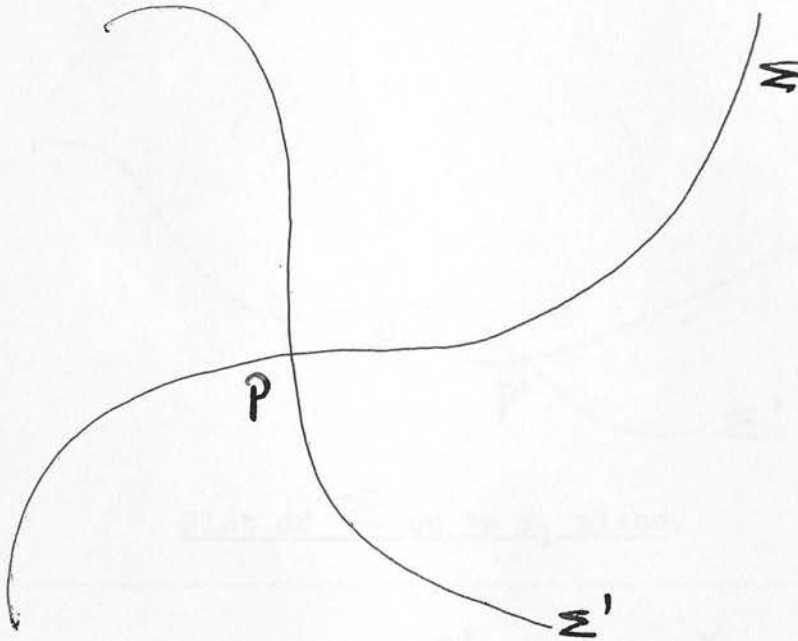


Fig. 14.



Plot of Σ on to z_1 plane.

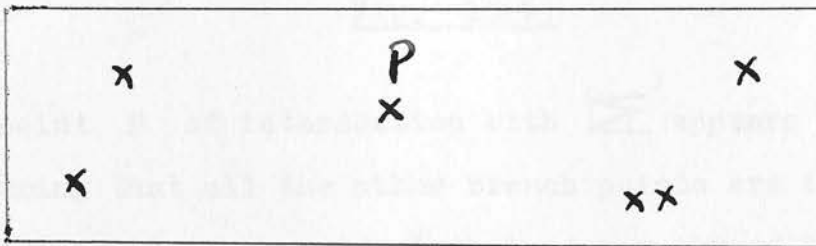
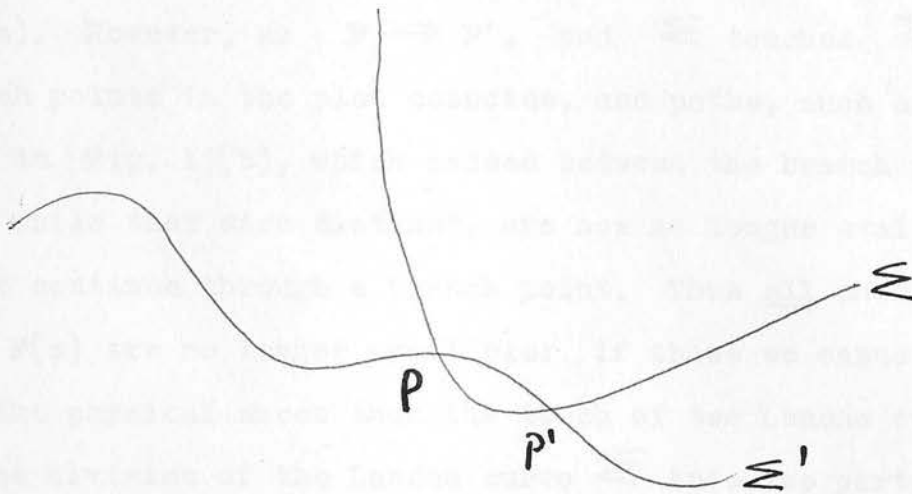


Fig. 15(a)

of the cuts in both z_1 and z_2 . Thus since Γ is plotted by AB, AB must lie between $z_1 = a$ and $z_1 = \infty$. Further the line which plots the intersection of the cut $b \leq z_2 \leq \infty$ with the Landau curve must also include AB.

Let us now consider the intersection of two Landau curves Σ and Σ' , and suppose that we are performing continuations of $F(z)$ on Σ , and that $F(z)$ is non-singular. The intersection with Σ' is a point. Thus, in the plot of Σ on to the z_1



Plot of Σ on to z_1 plane.

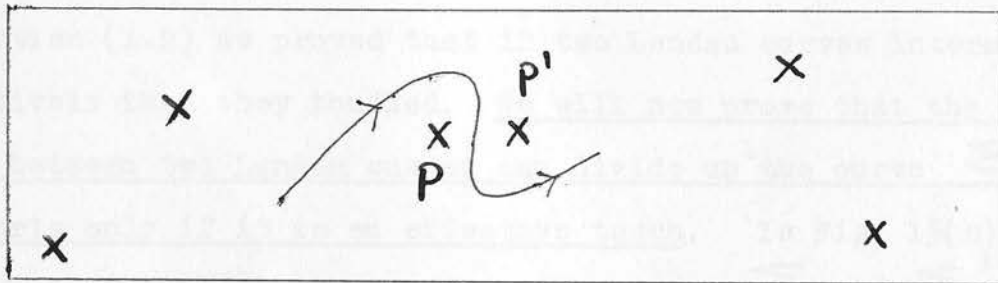


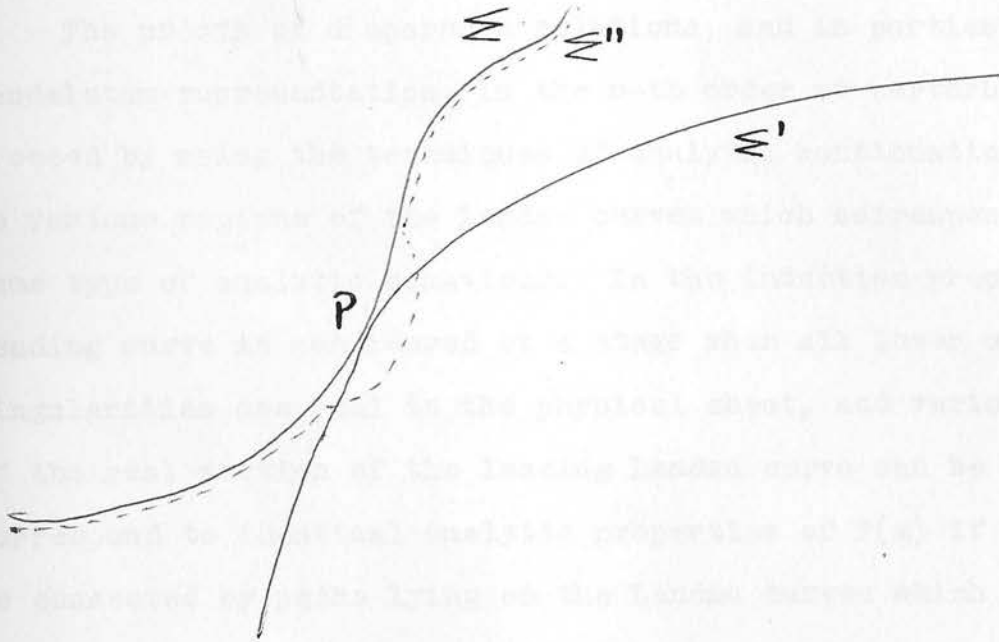
Fig. 15(b)

plane the point P of intersection with Σ' appears as a single point. Assuming that all the other branch points are isolated as shown in Fig. 15(a) it is evident that we may thread the branch points in any way we wish -- so reaching every sheet of $F(z)$. We may thus continue $F(z)$ on the Landau curve Σ as we wish without any change in the nature of $F(z)$. We conclude, then, that an ordinary intersection cannot divide the Landau curve up.

Now let us consider the configuration of Fig. 15(b) where the Landau curves Σ and Σ' have, locally, two intersections. This, while P and P' remain distinct points, is just the situation of

Fig. 15(a). However, as $P \rightarrow P'$, and Σ touches Σ' , the two branch points in the plot coincide, and paths, such as that sketched in (Fig. 15(b)), which passed between the branch points P and P' while they were distinct, are now no longer available since we cannot continue through a branch point. Thus all sheets of the function $F(z)$ are no longer available: if those we cannot reach include the physical sheet then the touch of two Landau curves has caused the division of the Landau curve Σ into two parts and we cannot continue from one to the other on Σ .

Clearly, then, touching of two Landau curves is very significant. In Section (1.2) we proved that if two Landau curves intersected effectively then they touched. We will now prove that the tangency point between two Landau curves can divide up the curve Σ into two parts only if it is an effective touch. In Fig. 15(c) we have drawn a touching situation between the curves Σ and Σ' . Construct a third curve Σ'' which coincides with Σ except near the point P of tangency, where it differs only very slightly from Σ . In the plot of Σ'' on to the z_1 plane the branch point P will have split up into two branch points and the plane will no longer be divided, corresponding to the fact that we may find a route past P on the curve Σ'' which keeps on the physical sheet. However, in doing this we have left the Landau curve Σ so that, in the α -plane, the coincident zeros of D , to which Σ corresponded, are slightly separated. If Σ' is not a lower order curve which meets Σ effectively then the slightly separated pair of zeros in the α -plane are never in the neighbourhood of an end point



Plot of Σ'' on to z_1 plane.

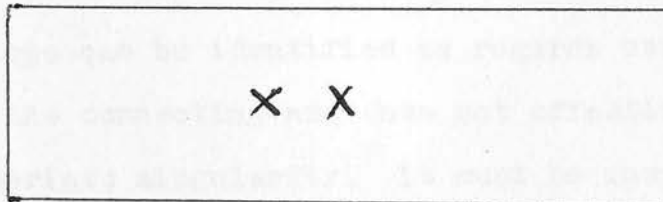


Fig. 15(c).

(continuity argument) and so no slipping over an end point (see Fig. 5) can occur. Thus when the path of continuation rejoins the zeros of D coincide without causing a pinch. On the other hand, if Σ' is an effective intersection, it is quite possible that when the path on Σ'' joins Σ again a singularity of the pinch type will appear.

In this way effective tangencies can divide up the Landau

curves into regions.*

The proofs of dispersion relations, and in particular the Mandelstam representation, in the n -th order of perturbation theory proceed by using the techniques of analytic continuation to connect up various regions of the Landau curves which correspond to the same type of analytic behaviour. In the induction process each leading curve is considered at a stage when all lower order singularities are real in the physical sheet, and various portions of the real section of the leading Landau curve can be shown to correspond to identical analytic properties of $F(z)$ if they can be connected by paths lying on the Landau curves which do not pass through effective intersections with lower order curves.

Outside the region of the crossed cuts the matter is trivial.

Otherwise two arcs** of given slope connected by a single arc of opposite slope can be identified as regards analytic behaviour provided that the connecting arc does not effectively intersect an arc of inappropriate singularity. It must be inappropriate because it is of lower order and so is assumed to have no attached complex singularities in the physical sheet, i.e. no arcs which are appropriately singular. The method of constructing a path between two arcs of given slope via adjacent unphysical sheets has been discussed by Landshoff, Polkinghorne and Taylor. Thus arcs of inappropriate

* Strictly the argument presented here applies only to appropriate singularity or non-singularity. However, the inappropriate case is always the case which is appropriate in some unphysical sheet, and the argument is precisely the same.

** By 'arc' we usually mean a portion of the real section of a curve.

singularity are of fundamental importance* in the methods of analytic continuation of $F(z)$, and it is to a discussion of their properties that the following paragraphs are devoted.

In general, arcs of inappropriate singularity can have no horizontal or vertical tangents.** Landshoff, Polkinghorne and Taylor say that a change in the sign of the gradient of the real section of a Landau curve lying wholly inside a region where $F(z)$ has cuts in both z_1 and z_2 , implies a change in singularity type from appropriate to inappropriate (or vice versa). The reason for this is that, as we continue on Σ from one side to the other of the point of horizontal or vertical tangency, the relative sign of the imaginary parts of z_1 and z_2 changes because we have gone through either the z_1 or the z_2 cut into an adjacent unphysical sheet. If at the outset our singularity was an appropriate one, we now have $F(z)$ appropriately singular in an adjacent unphysical sheet. By the first result of the present section this implies that $F(z)$ is inappropriately singular at the corresponding points of the physical sheet. It now follows that an arc which is only inappropriately singular cannot possess horizontal or vertical tangents unless there exist complex singularities in the physical sheet (a circumstance forbidden by the induction hypothesis). This is because, if it did possess such tangents, a contradiction arises: for, by assumption, the arc beyond the tangency point is free from complex sprouts into the physical sheet and is thus appropriately non-singular: then, by the above argument, the

* Such arcs define the boundaries of the Mandelstam spectral functions.

** This statement is not intended to include inflectional tangents.

original arc must have been inappropriately non-singular which is a contradiction.

The argument presented in the last paragraph is blatantly false in the case when the gradient change is accompanied by an effective intersection with a lower order singular curve. In the appendix a graph of the vertex type is illustrated in Fig. 8: it lies entirely within crossed cuts, but it is touched by the normal threshold curves at precisely those points at which the gradient changes; thus at the tangency point the relative sign of the imaginary parts of z_1 and z_2 changes on the complex part of Σ because of the gradient change; but also because of the effective intersection, a singularity occurring with the imaginary parts of z_1 and z_2 of like sign disappears and is replaced by one with unlike signs (or vice versa); the net effect is that the singularity remains appropriate despite two changes of gradient inside the crossed cut region.

In the case of the elastic scattering problem, there is no serious difficulty arising from this type of behaviour; in the remainder of this section we will assume, except where explicit statement of the contrary is made, that arcs of inappropriate singularity have no horizontal or vertical tangents.

Arcs, singular only in the inappropriate sense cannot lie across a Landau curve which corresponds to a normal cut and so pass out from the crossed cut region. By drawing a plot of the curve on to the z_1 plane it is clear that, on any curve such as Σ_1 , whose real section is drawn in Fig. 16(a), a continuation

path cannot be blocked by the single intersection with the lower order curve: thus we may conclude that $F(z)$ is singular in the inappropriate

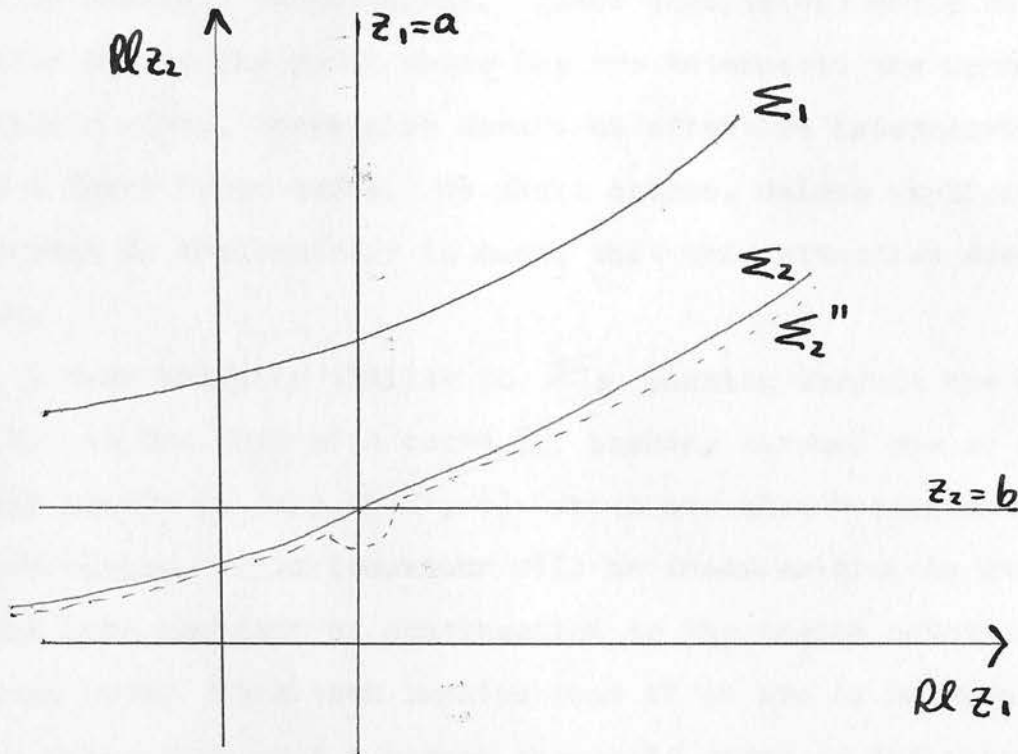


Fig. 16(a)

sense to the left of $z_1 = a$: this, however, also implies appropriate singularity because we are outside the crossed cut region. This contradicts the induction hypothesis, so we must exclude curves such as Σ_1 from the possible types having arcs of inappropriate singularity. On the curve Σ_2 it is not immediately clear that we cannot be blocked, because two branch points coincide in the plot of Σ_2 on to the z_1 plane. By the device already employed we can choose a slightly different curve Σ_2'' which is not blocked: the intersection is not effective, and so, by the usual argument we may

deduce the behaviour of $F(z)$ on Σ_2 outside the crossed cut region.* Just as with Σ_1 , we find that curves such as Σ_2 are, in general, inadmissible. These conclusions would be invalid if, at the point where the arc intersects the normal threshold curve, there also occurs an effective intersection with a lower order curve. We shall assume, unless explicit statement to the contrary is made, that this situation does not arise.

A case which is similar to Σ_2 passing through the point (a, b) , is the case of a curve Σ passing through one or other of the points (a, ∞) , (∞, b) which are also coincidences of branch points. This behaviour will be inadmissible, in general, by the same argument of continuation to the region outside the crossed cuts. This then implies that if an arc of inappropriate singularity does meet a normal threshold curve at infinity it meets it effectively, and this in turn gives us that the directions of the normals at the intersection is the same on both curves.

In order that this latter deduction should be true we must give a reason for excluding the possibility of the type of behaviour

* It can be argued that if an arc of inappropriate singularity ever meets a normal threshold effectively it does so at infinity. This is because, if it meets a normal threshold effectively, it also meets effectively all curves of intermediate order obtained by setting equal to zero any subset of the a_i which actually vanish at the intersection. The vertex curves which occur in the elastic scattering problem are parallel to the normal thresholds, and since they are the curves of second lowest order, the result follows because parallel lines meet at infinite points.

for Σ which is illustrated in Fig. 16(b); because, in the absence of an effective intersection, there is no way of continuing to that part of the real plane outside the crossed cut region by paths on Σ .

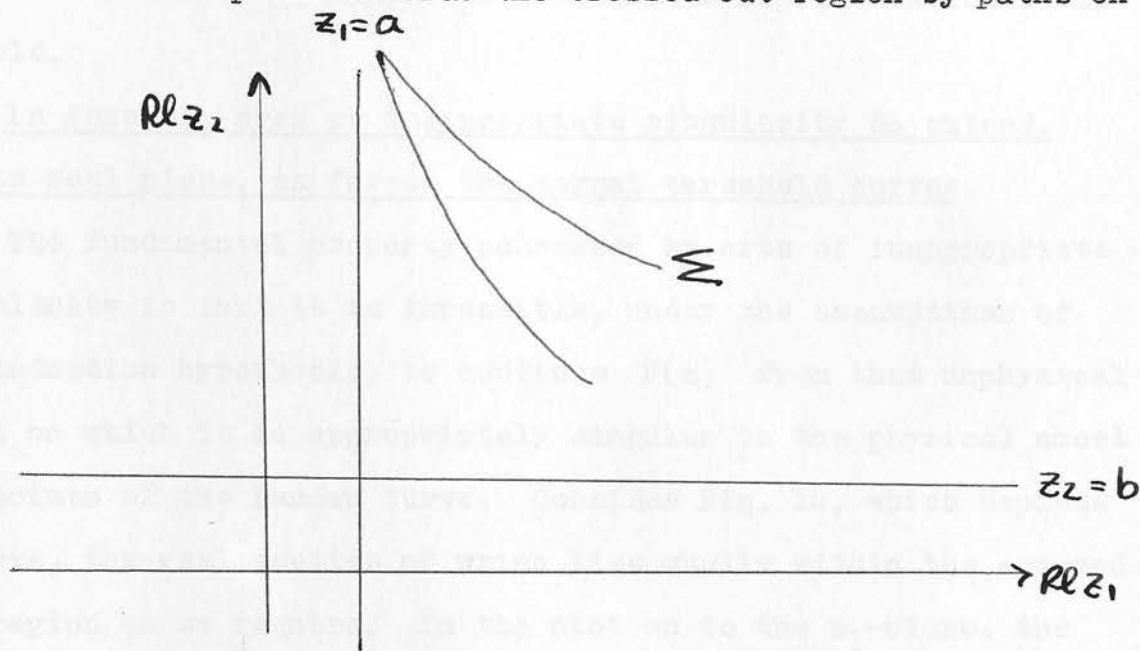


Fig. 16(b).

It was pointed out to the present author by Professor N. Kemmer, that, at cusp-like behaviour of this sort on a Landau curve, it was possible to find, in the neighbourhood of the cusp, points both appropriate and inappropriate to the arcs which formed the cusp.* Thus the induction hypothesis could be violated by such behaviour and so we should exclude such curves from consideration. However, it was very clear that algebraic curves could possess such behaviour and we could see no good reason for supposing that at no stage in the induction such a curve appeared as the leading Landau curve. We set aside this problem in the naive hope that physical situations would not necessitate considerations of cusp-like behaviour of Landau

* See Section (1.5).

curves. Subsequently, however, Eden, Landshoff, Polkinghorne and Taylor have given an explicit example which involves cusp-like behaviour: in their example the Mandelstam representation fails to hold.

In general, arcs of inappropriate singularity do extend, in the real plane, as far as the normal threshold curves.

The fundamental property possessed by arcs of inappropriate singularity is that it is impossible, under the assumptions of the induction hypothesis, to continue $F(z)$ from that unphysical sheet on which it is appropriately singular to the physical sheet via points of the Landau curve. Consider Fig. 14, which depicts a curve, the real section of which lies wholly within the crossed cut region as we require. In the plot on to the z_1 -plane, the real section AB has the properties $a \leq A, B$ and $A, B \leq \infty$. For our argument we require to assume that the branch points actually drawn (marked by crosses in Fig. 14) are the only relevant ones. It is clear that, in the case illustrated, both cuts can be passed through separately, and every sheet of $F(z)$ is accessible on all curves satisfying $a < A, B$ and $A, B < \infty$. Thus arcs, such as those sketched in Figs. 16(c) and 16(d), will certainly not do as arcs of inappropriate singularity. The situations in which our route of continuation on the Landau curve is definitely blocked are those in which A and B coincide with $z_1 = a$ and $z_1 = \infty$ (which also forces them into coincidence with $z_2 = b$ and $z_2 = \infty$). This proves the stated result.

All this information now tells us that arcs of inappropriate

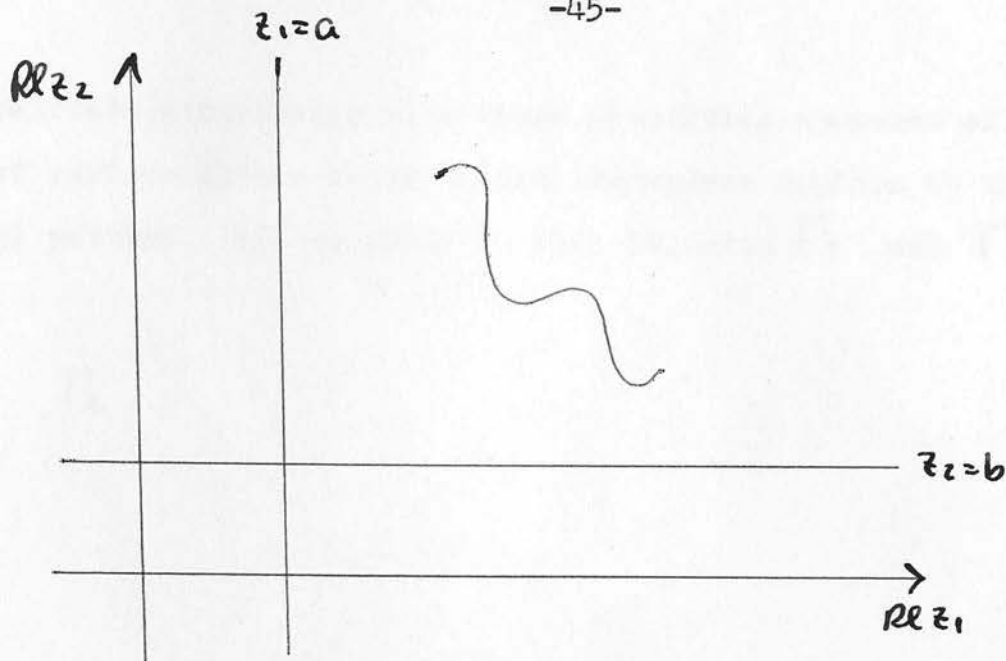


Fig. 16(c).

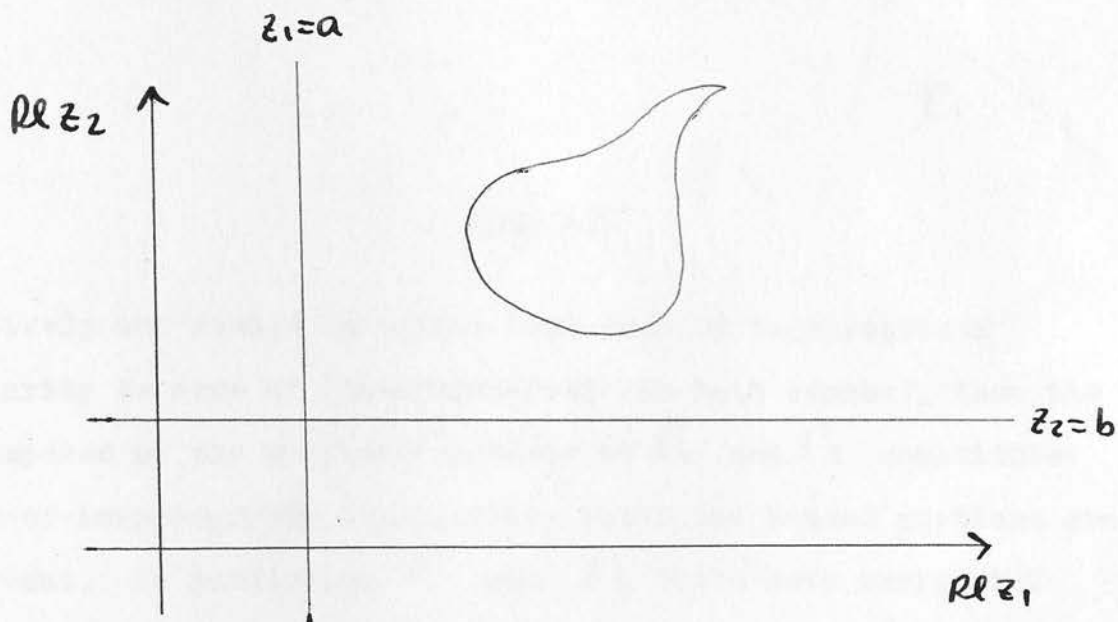


Fig. 16(d).

singularity are, in general, arcs of negative slope, with asymptotes parallel to the axes of z_1 and z_2 , which lie wholly inside the region of the crossed cuts. We can, in fact, have an arc of

inappropriate singularity with these properties composed of several arcs of various orders which do not themselves conform to the general pattern. If, as shown in Fig. 17, arcs Γ_1 and Γ_2 touch

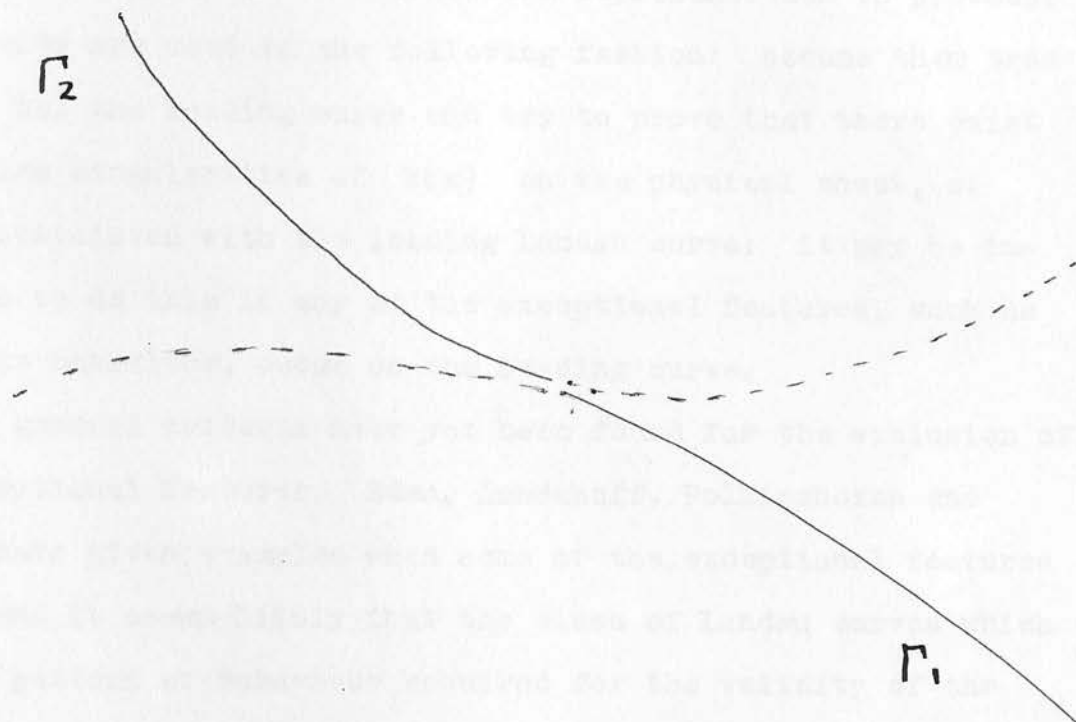


Fig. 17.

effectively and change in nature from arcs of inappropriate singularity to arcs of non-singularity (in both senses), then the arc composed of the undotted portions of Γ_1 and Γ_2 constitutes an arc of inappropriate singularity, while the dotted portions are irrelevant. In particular, Γ_1 and Γ_2 could have horizontal (or vertical) non-inflectional tangents at the intersection while the arc which is essentially the arc of inappropriate singularity does not.

In those cases when our general theorems are invalid, the feeling is that it is not a terribly important matter unless the

situation persists for a whole range of values of the external masses of the problem.

These properties, which we have discussed in detail, permit the inductive proof of the Mandelstam representation to proceed. The results are used in the following fashion: assume them true for all but the leading curve and try to prove that there exist no complex singularities of $F(z)$ on the physical sheet, at points associated with the leading Landau curve: it may be impossible to do this if any of the exceptional features, such as cusp-like behaviour, occur on the leading curve.

No general criteria have yet been found for the exclusion of the exceptional features. Eden, Landshoff, Polkinghorne and Taylor have given examples when some of the exceptional features arise* and it seems likely that the class of Landau curves which fit the pattern of behaviour required for the validity of the inductive proof will not include all interesting physical cases.

(1.5) The Scattering Amplitude and the Proof of Multiple Dispersion Relations

The work of Eden,⁽²⁾ of Landshoff, Polkinghorne and Taylor, and Eden, Landshoff, Polkinghorne and Taylor has done much to promote the understanding of the analytic properties of collision amplitudes, and, before the discovery of acnodes (isolated points), crunodes (double points), and cusps on the Landau curves a great

* In the example quoted here, acnodes, crunodes, and cusps, persist for a wide range of values of the external masses.

deal had been done to imbue field theorists with confidence in the value of dispersion relations, and, in particular, in the truth of the Mandelstam conjecture for the elastic scattering amplitude. Basically their methods of proof consisted in exploiting a limited knowledge of proven dispersion relations and a knowledge of regions of regularity of the amplitudes, by means of the powerful method of analytic continuation. The Landau curves themselves very often provide suitable vehicles for the continuations, because, starting from a point at which the analytic behaviour is known, we can move freely on the Landau curves provided we take proper account of the set of points at which $F(z)$ may change its nature -- thus far the only members of this set were unknown to the author at the time when the present work was in hand and they are associated with the exceptional features such as acnodes, crunodes, and cusps which were discovered by Eden, Landshoff, Polkinghorne and Taylor. Briefly, the mechanism is that, in the α -plane, instead of two zeros of D coinciding and causing a pinch -- three zeros coincide: different modes of continuation in the z -plane lead to different pairings of the zeros in the α -plane. As (1) and (2) (see Fig. 18) approach the contour they pinch it. As (2) and (3) do so they coincide harmlessly. At a triple coincidence singular and non-singular behaviour may interchange corresponding to (2) changing from its association with (1) to an association with (3) (or vice versa).*

The discovery of these new members of the set of points where $F(z)$ may change its nature are associated with the appearance in the physical sheet of complex singularities, and must be regarded as a major set back in the progress towards the goal of a new starting point for the theory of elementary particle phenomena. If, as

* Such a mechanism can give rise to cusp-like behaviour of a Landau curve.

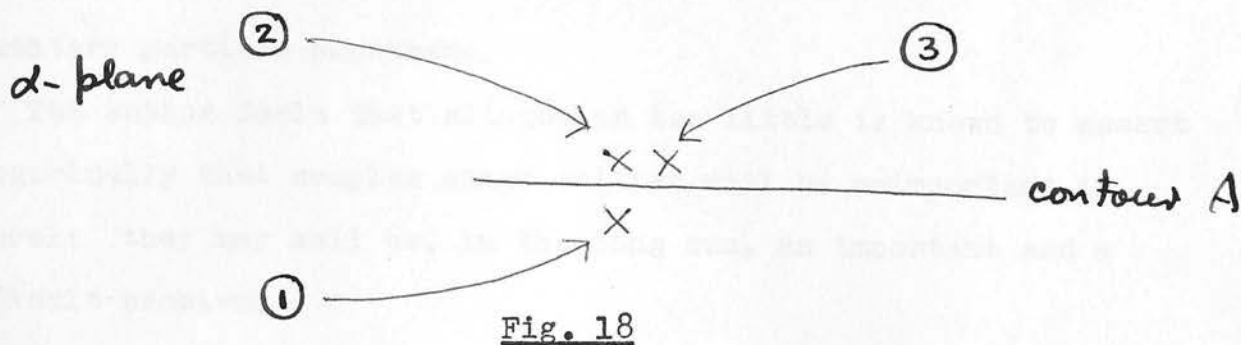


Fig. 18

the work of Polkinghorne⁽⁷⁾ suggests, these features occur not only in perturbation theory but in any unitary theory the matter is very serious indeed and requires much further investigation. One hopes that methods can, and will be evolved to cope with these new features of the scattering amplitude.

Mandelstam⁽⁶⁾, himself, does not seem to consider the matter a disaster, and in fact believes that there exists no difficulty, in principle, in fitting complex singularities into the existing scheme.

The reason for this present optimism is that one hopes that the physical data, in which one is interested, are dominated by singularities of the well-understood type, i.e. that integrals over complex branch cuts will lead to negligible contributions. In a sense, current experimentation confirms this hypothesis because reasonable results have been obtained by calculations based on the assumption that processes are dominated by a small number of singularities. This sort of outlook tends to downgrade dispersion theory to a mere approximation scheme and hits hard the attitude of mind which has been seeking in the study of analytic properties of collision amplitudes some hint of a fundamental understanding of

elementary particle phenomena.

The author feels that altogether too little is known to assert categorically that complex singularities will be unimportant in general: they may well be, in the long run, an important and a difficult problem.

Encouraged by the belief that existing techniques had had some measure of success in coping with the elastic scattering problem, the author abandoned this study to embark on an investigation of the less well understood processes of inelastic scattering.

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CHAPTER 2

INELASTIC PROCESSES

(2.1) The General Problem

Despite several notable attempts to establish results in perturbation theory for graphs of arbitrary complication, successes, though encouraging in many ways, have been only partial, and it is true to say that only the two, three, and four point single loop graphs are completely understood: as these are the simplest graphs corresponding to the various processes, an understanding of them is clearly vital for the understanding of more complicated graphs and thus of the total amplitude concerned. The next graph in logical succession is the five-point graph: this graph is exceedingly important for a variety of reasons which will be elaborated in Section (2.2). It is, unfortunately, much more complicated to describe than graphs for processes involving fewer particles because the number of independent variables for the problem increases rapidly with the total number of external particles interacting. In this connection the author has proved the following theorem: in a space of dimensionality ν , the number of independent scalar product variables which can be formed from n vectors of arbitrary direction but constant length whose sum is zero is given by the formula

$$N = \frac{1}{2}n(n-3) - \frac{1}{2}(n-\nu-1)(n-\nu)\theta(n-\nu-1) \quad (2.1.1)$$



Let us suppose that we have n vectors, as yet unrestricted, p_1, p_2, \dots, p_n , in a space of dimensionality ν , $n \geq \nu$, and that we define scalar products s_{ij} as follows:

$$S_{ij} = p_i \cdot p_j = \sum_{\alpha, \beta}^{\nu} g_{\alpha\beta} p_{\alpha i} p_{\beta j} \quad (2.1.2)$$

where $g_{\alpha\beta} = g_{\beta\alpha}$ and $\det(g_{\alpha\beta}) \neq 0$.

In a matrix notation we simply write equation (2.1.2) in the form:

$$S = \tilde{P} G P \quad (2.1.3)$$

If we denote the rank of a matrix by the symbol r , we have $r(G) = \nu$ and, in general, $r(P) = \nu$, so that, in general, $r(S) = \nu$.

It is possible to invent an orthogonal matrix O such that the symmetric matrix S can be written as:

$$S = \tilde{O} \Lambda O \quad (2.1.4)$$

where Λ is the diagonal matrix given by

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n) \quad (2.1.5)$$

Since $r(S) = \nu$ we may write

$$\lambda_{\nu+1} = \lambda_{\nu+2} = \dots = \lambda_n = 0 \quad (2.1.6)$$

and because of this identity among the latent roots of S the

of independent parameters in the matrix O is

$\frac{1}{2}n(n-1) - \frac{1}{2}(n-\nu)(n-\nu-1)$. Thus the matrix S depends on $\frac{1}{2}n(n-1) - \frac{1}{2}(n-\nu)(n-\nu-1) + \nu$ independent parameters.

If, additionally, the sum of the vectors p_i is zero, i.e.

$$\sum_{i=1}^n p_i = 0 \quad (2.1.7)$$

we may simply consider a subset of the p_i , say the set p_1, \dots, p_{n-1} , and the number of independent scalar products which can be constructed is

$$N = \frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-\nu-1)(n-\nu-2) + \nu \quad (2.1.8)$$

where $n \geq \nu + 1$.

Included amongst the scalar products are the lengths of the vectors p_i so that if we do not wish to include these we must diminish N by n .

It is also important to discuss the case when the number of vectors p_i are insufficient to span the space, i.e. $n < \nu$. In this case let us first take as independent the vectors p_1, \dots, p_n and so $r(P) = n$, and consequently $r(S) = n$. S will be a general symmetric matrix having $\frac{1}{2}n(n+1)$ independent elements. If, in addition, condition (2.1.7) holds, and we discount the lengths of the vectors, the number of independent parameters becomes:

$$N = \frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3) \quad (2.1.9)$$

where we must have $3 \leq n < v + 1$.

To sum up, then, for any set of vectors p_1, \dots, p_n , in a space of v dimensions there can be constructed N scalar products where

$$N = \frac{1}{2} n(n+1) - \frac{1}{2} (n-v)(n-v+1) \theta(n-v) \quad (2.1.10)$$

If, also, condition (2.1.7), which requires the vectors to have vanishing sum, is imposed, N is given by

$$N = \frac{1}{2} n(n-1) - \frac{1}{2} (n-v-1)(n-v) \theta(n-v-1) \quad (2.1.11)$$

Finally, if we further impose the condition that the p_i have constant length we obtain our final result

$$N = \frac{1}{2} n(n-3) - \frac{1}{2} (n-v-1)(n-v) \theta(n-v-1) \quad (2.1.1)$$

where $n \geq 3$.

In the case of physics $v = 4$, the values of N given by formula (2.1.1) corresponding to $n = 3, 4, 5$, and 6 , are $N = 0, 2, 5$, and 8 ; and in general

$$N = 3n - 10, \quad n \geq 4 \quad (2.1.12)$$

A completely different proof of the same result (2.1.12) has been given by Chan Hong-Mo⁽²⁾: in his original manuscript incorrect results were obtained: this, however, has been modified after private communication with present author:

reference to the above given proof is made in the second of the two references quoted.

The point which emerges is that the number of independent variables which describe the problem increases rapidly with n and this, in itself, presents new problems.

Incidentally, the result (2.1.12) leads to an immediate generalisation of some work of Okun and Rudik⁽⁵⁾ who were responsible for developing the elegant method of solving the Landau equations using dual-diagrams: they establish, in simple cases, criteria for solving the Landau equations. Their method is applicable to any arbitrary graph by means of formula (2.1.12) in the following manner:

The law of four-momentum conservation when applied at each vertex of a Feynman graph leads to n' equations involving the external and internal four-momentum vectors (n' is the total number of vertices in the graph). Take one of these equations as expressing the overall conservation law (2.1.7) and the remaining $n' - 1$ to be any other set of relationships between the four-vectors. Thus we may suppose that we have at our disposal $n' - 1$ vector equations for the $4L$ components of the vectors q_i (taking L to be the number of internal vectors q_i) and the $3n - 10$ invariants of the problem (assuming that there are n external particles interacting). Moreover the components of the q_i are restricted by the fact that their lengths are fixed by equations (1.1.4). Thus, altogether, $4n' - 4 + L$ scalar equations can be written down for the $3n - 10 + 4L$ unknowns.

There are, of course, also the Landau loop equations

(1.1.5) which involve the vectors q_i , and the normalisation condition on the α 's (1.1.3). Of these there are ℓ vector equations and 1 scalar equation. Since ℓ , for an arbitrary Feynman diagram, can be shown to be given by

$$\ell = L - n' + 1 \quad (2.1.13)$$

there exist $4L - 4n' + 5$ scalar equations which involve the L α 's.

It is thus clear that it may be impossible to determine all the α 's, and those which remain indeterminate number $4n' - 3L - 5$. If this number is zero or negative we can determine the α 's and there remain equations which we can use to help to determine the components of the q_i and the invariants. Thus when $4n' - 3L - 5 \leq 0$ we have $3L - 4n' + 5$ extra equations; the number of components of the q_i together with invariants is $3n - 10 + 4L$ so that, in the end, the number which will remain undetermined is:

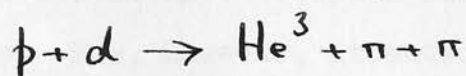
$$\begin{aligned} & (3n - 10 + 4L) - \left[(4n' - 4 + L) + (3L - 4n' + 5) \right] \\ & = 3n - 11 \end{aligned}$$

Thus, then, since there are $3n - 10$ invariants there must exist a relationship between them: this relation is a Landau curve.

This result probably represents part of a very important existence theorem.

(2.2) The Five-Point Problem

The ultimate aim of theoretical physics is to understand and to describe natural phenomena, and, although the present dissertation is several stages removed from things practical, the common occurrence of five-point processes must be the prime justification for the present chapter. As mentioned in the introduction, the pion-pion interaction is, as a consequence of dispersion theory, the subject of much active present day experimentation. In this connection a series of experiments are being carried out at the Lawrence Radiation Laboratory in California on the processes



Experiments on the decay of a deuteron and an antideuteron into three pions is also of current interest in connection with the study of deuteron form factors.

There are also, however, several compelling and important theoretical reasons for the study of five-point processes, not simply for their own sakes, but because they are related to other important processes such as four-point interactions - elastic scattering.

Unitarity plays a central role in the study of scattering. The unitarity of the S matrix, $SS^\dagger = 1$, implies that the invariant amplitude T satisfies the equation

$$2\text{Im} \langle a | T | b \rangle = \sum_n \langle a | T | n \rangle \langle n | T^\dagger | b \rangle \quad (2.2.1)$$

where energy is conserved in the intermediate states.

Diagrammatically one usually writes equation (2.2.1) as shown in Fig. 19.

$$\text{Im } a \left\{ \begin{array}{c} \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \end{array} \right\} b = \begin{array}{c} \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \end{array} + \begin{array}{c} \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \\ \text{---} \bigcirc \text{---} \end{array} + \dots$$

Fig. 19.

In this connection, up to the present time, two-particle intermediate states have occupied fully the time and energies of theorists. Now, however, attention is being turned to the consideration of three particle intermediate states and such states require information about production processes.

Certain workers in the subject have recently been studying the amplitude, not only on its physical sheet, but on sheets of the function which can be reached by analytically continuing through the branch cut corresponding to two-particle intermediate states (the first normal threshold^{*}) at energies below the threshold for the production of three particles. There are several reasons for such studies. Firstly, the anomalous threshold singularities which do not correspond to the possibility of any new competing process, have their origin in unphysical sheets of the function and appear in the physical

* Singular behaviour clearly arises whenever the values of n permit a new process to become energetically possible: such singularities are called normal thresholds.

sheet only for certain values of the masses of the problem. Secondly, one method of approach to the theory of unstable particles looks to unphysical sheets of a function for the explanation of certain resonance effects in the physical sheet due to poles of the amplitude close to, but just off, the physical sheet. Now in such investigations, continuations have been made only through the cuts running from the first normal threshold to the first inelastic threshold; clearly the lack of knowledge of production processes is holding up the possibilities of continuing through three-particle and higher branch lines. Thus a knowledge of production processes would assist greatly in the elastic scattering problem.

Conversely, of course, postulates about the analytic structure of the scattering amplitude -- for example the Mandelstam conjecture -- could lead, via unitarity, to predictions about production amplitudes. This problem appears very complicated and the best hope seems to be to attack production processes directly.

Finally, for purely academic reasons, the study of the simplest production graph -- the five-point loop -- is of great interest. This is because the loops of order four and less are well understood and it has been shown by Brown⁽¹⁾ that all single loop graphs of order six or more are expressible algebraically in terms of lower order diagrams. Thus if one could describe the five-point loop, all single loop diagrams could be discussed, and single loop diagrams as a class would represent known quantities. Although this would be

aesthetically pleasing it could not be regarded as a major step forward unless more complicated graphs could also be so treated. Nevertheless it should be realised that the class of single loop diagrams are models for a much less restrictive class of processes than it might seem at first glance. In Fig. 20 is drawn a scattering graph of the simple loop type with four internal particles of masses m_1 , m_2 , m_3 and m_4 .

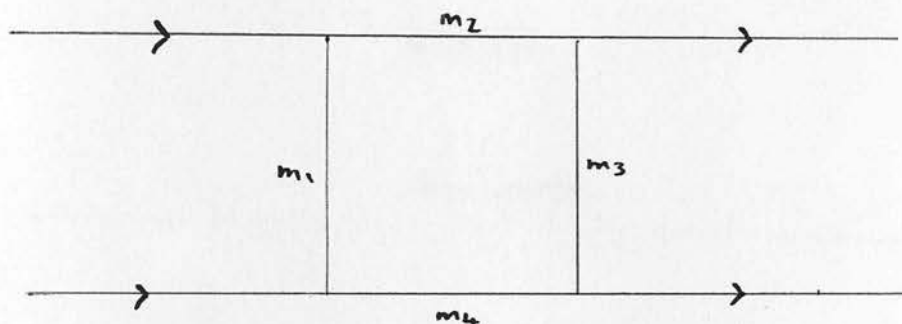


Fig. 20.

Such a graph can be utilised to describe processes involving certain more complicated intermediate states. In the elastic scattering of nucleons on nucleons, for example, any process of the type depicted in Fig. 21 is adequately represented, in our version of perturbation theory, by the diagram of the type of Fig. 20 as shown in Fig. 22. The solid lines represent nucleons of mass M_N , while the broken lines represent pions of mass M_π .

From a physical standpoint one would expect intermediate states of two nucleons and several pions to give very important contributions to the scattering amplitude for nucleon-nucleon

scattering: thus it is very encouraging that such diagrams can be treated in an elementary fashion in perturbation theory.

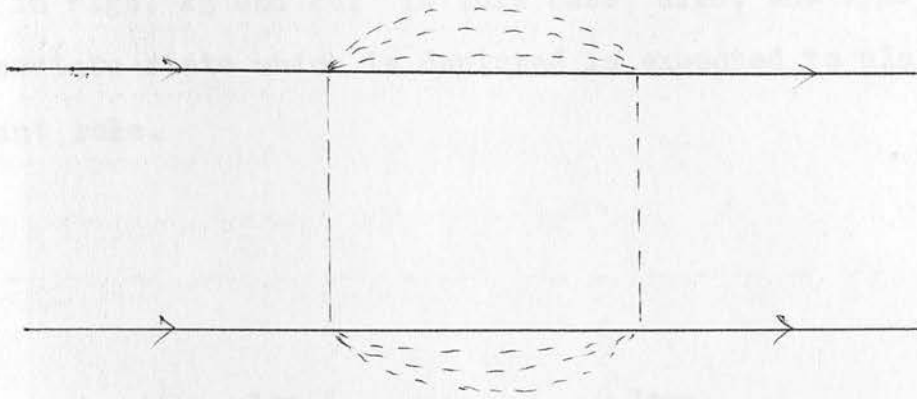


Fig. 21.

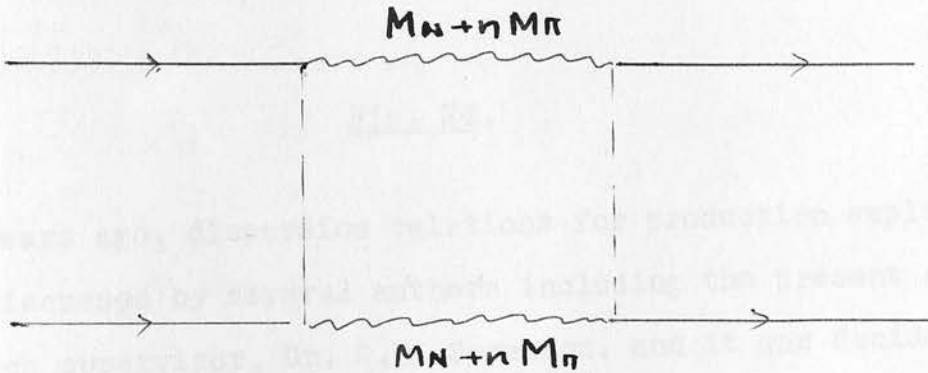


Fig. 22.

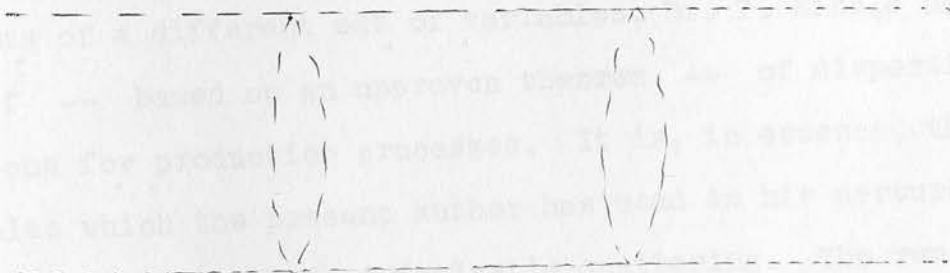


Fig. 23.

A similar example, drawn from pion-pion scattering, is shown in Figs. 23 and 24; in this case, also, the type of intermediate state which is depicted is expected to play a dominant role.

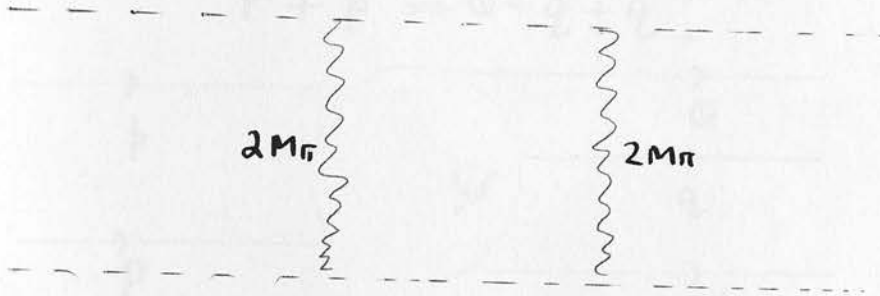


Fig. 24.

Some years ago, dispersion relations for production amplitudes were discussed by several authors including the present author's research supervisor, Dr. G.R. Screaton, and it was decided that it would be a valuable exercise to study such matters in the perturbation theoretic framework. As we shall see, the scalar product variables usually employed in dispersion theory lead to difficulties which, as yet, have not been surmounted. However, in terms of a different set of variables, Dr. T. Kibble has given a proof -- based on an unproven theorem -- of dispersion relations for production processes. It is, in essence, these variables which the present author has used in his perturbative approach to the problem of inelastic scattering. The remainder of this chapter gives an account of this investigation, the results of which were obtained independently of those of Kim⁽³⁾ which have recently been published,

(2.3) Kinematical Considerations^{*}

Consider the process $\pi + N \rightarrow \pi + \pi + N$ which is depicted in Fig. 25. Let the incoming nucleon have four-momentum P and the outgoing one Q : also take the incoming muon to have four momentum p and the outgoing ones q and q' .

Energy-momentum conservation implies that

$$P + p = Q + q + q' \quad (2.3.1)$$

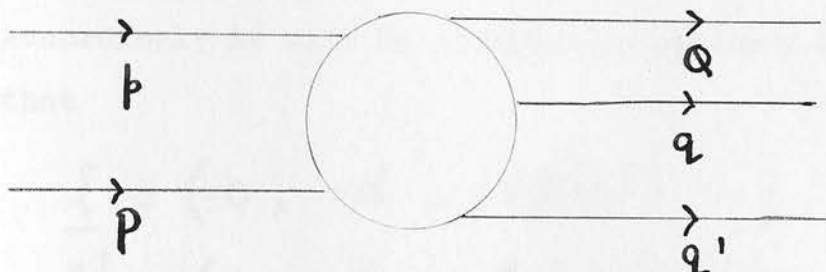


Fig. 25

If we choose the coordinate system such that the vectors P, Q, p, q, q' can be written

$$\begin{aligned} P &= (\sqrt{M_N^2 + \Delta^2}, 0, 0, \Delta) & Q &= (\sqrt{M_N^2 + \Delta^2}, 0, 0, -\Delta) \\ p &= (\omega, \underline{p}) & q &= (v\omega, \underline{q}) \\ & & q' &= (v'\omega, \underline{q}') \end{aligned} \quad (2.3.2)$$

then equation (2.3.1) implies that

$$(\sqrt{M_N^2 + \Delta^2} + \omega, \underline{p} + \underline{\Delta}) = (\sqrt{M_N^2 + \Delta^2} + (v+v')\omega, \underline{q} + \underline{q}' - \underline{\Delta}) \quad (2.3.3)$$

It then follows that

$$v + v' = 1 \quad \text{and} \quad \underline{q} + \underline{q}' = \underline{p} + 2\underline{\Delta} \quad (2.3.4)$$

We define four vectors ξ, ξ' as follows

* As stated in Section (1.1) any dependence on charge or spin will be ignored.

$$\left. \begin{aligned} \underline{\xi} &= \underline{v} \underline{p} - \underline{q} = (0, \underline{v} \underline{p} - \underline{q}) = (0, \underline{\xi}) \\ \underline{\xi}' &= \underline{v}' \underline{p} - \underline{q}' = (0, \underline{v}' \underline{p} - \underline{q}') = (0, \underline{\xi}') \end{aligned} \right\} \quad (2.3.5)$$

Clearly then, equation (2.3.4) implies that

$$\underline{\xi} + \underline{\xi}' + 2 \underline{\Delta} = 0 \quad (2.3.6)$$

Accordingly it will be possible to choose a Lorentz frame such that

$$\left. \begin{aligned} \underline{\xi} &= (0, -d, -(\Delta+c)) \\ \underline{\xi}' &= (0, d, -(\Delta-c)) \end{aligned} \right\} \quad (2.3.7)$$

It is now fairly evident that \underline{p} , \underline{q} , \underline{q}' can be expressed in terms of the parameters Δ , c , d , \underline{v} , \underline{v}' , and the meson energy w . This is in agreement with the general theorem of (2.1) since \underline{v} , \underline{v}' are related by equation (2.3.4). One finds that the components of \underline{p} are given as follows:

$$p_1 = \lambda(w) = \sqrt{w^2 - \varepsilon^2} \quad (2.3.8)$$

where

$$\varepsilon = \sqrt{M_\pi^2 + p_2^2 + p_3^2}$$

$$p_2 = \frac{1}{4vv'\Delta d} \left\{ (v-v')\Delta [(1+vv')M_\pi^2 + d^2 - c^2 + \Delta^2] + c [(1-vv')M_\pi^2 + d^2 + c^2 - \Delta^2] \right\} \quad (2.3.9)$$

$$p_3 = \frac{-1}{4vv'\Delta} \left\{ v(\Delta-c)^2 + v'(\Delta+c)^2 + (1-vv')M_\pi^2 + d^2 \right\} \quad (2.3.10)$$

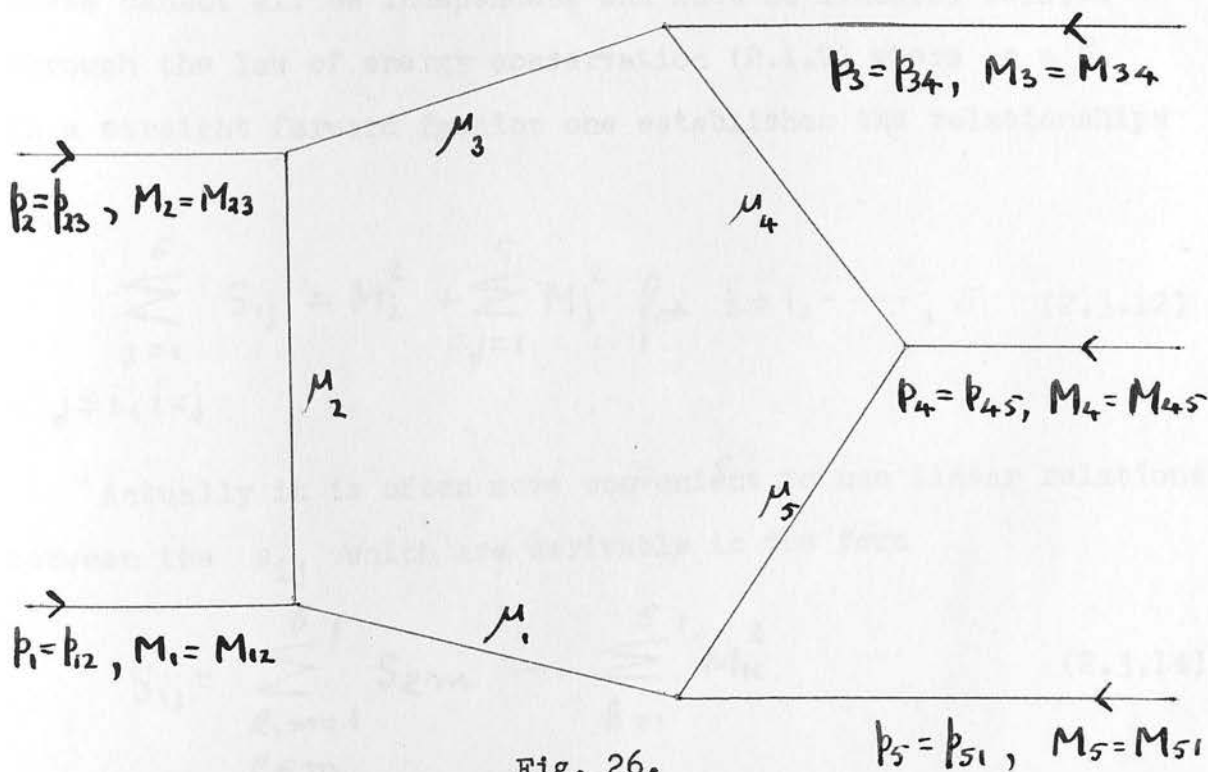
The components of the vectors \underline{q} , \underline{q}' are then expressible in terms of the parameters via equations (2.35) and (2.37).

Finally one obtains

$$\left. \begin{aligned} P &= (\sqrt{M_N^2 + \Delta^2}, 0, 0, \Delta) \\ p &= (\omega, \lambda(\omega), p_2, p_3) \end{aligned} \right\} \quad \left. \begin{aligned} Q &= (\sqrt{M_N^2 + \Delta^2}, 0, 0, -\Delta) \\ q &= (v\omega, v\lambda(\omega), v p_2 + d, v p_3 + \Delta + c) \\ q' &= (v'\omega, v'\lambda(\omega), v' p_2 - d, v' p_3 + \Delta - c) \end{aligned} \right\} \quad (2.3.11)$$

These expressions are real in the physical region $\omega \geq \varepsilon$. It is in terms of the variable ω that one hopes to write dispersion relations for the amplitude describing the production process.

We now set out, explicitly to study the single loop graph shown in Fig. 26, and we must first establish our notation



Formally we measure all our four-momenta as ingoing. We label our internal lines by a single label i ; for example, the mass of an internal particle in the i -th internal line is μ_i . We label our external lines by means of the labels of the internal lines which they meet: for example the four momentum of the line which meets the i -th and $i+1$ -th internal lines is $p_{i, i+1}$, and its mass is $M_{i, i+1}$ (modulo 5). It is, however, often unambiguous, and indeed more convenient, to drop the second suffix altogether, and we shall do this whenever possible. The complete scheme is shown in Fig. 26. Notice that, comparing Fig. 26 with Fig. 25 that

$$p_1 = P, \quad p_2 = p, \quad p_3 = -Q, \quad p_4 = -q, \quad p_5 = -q' \quad (2.3.12)$$

We now define our scalar products according to the rule $s_{ij} = (p_i + p_j)^2$ and it is immediately evident that there are ten such entities. According to the general theorem of (2.1) these cannot all be independent and must be linearly related through the law of energy conservation (2.1.7) where $n = 5$. In a straight forward fashion one establishes the relationships

$$\sum_{\substack{j=1 \\ j \neq i, i < j}}^5 s_{ij} = M_i^2 + \sum_{j=1}^5 M_j^2 \quad \text{for } i=1, \dots, 5 \quad (2.3.12)$$

Actually it is often more convenient to use linear relationships between the s_{ij} which are derivable in the form

$$s_{ij} = \sum_{\substack{l, m=1 \\ l < m}}^5 s_{lm} - \sum_{k=1}^5 M_k^2 \quad (2.3.14)$$

where, in the summations, none of ℓ , m , and k are permitted to take the values i or j . Of the set (2.3.14) there are clearly ten so that these equations, unlike equations (2.3.13), do not form an independent set of linear relations. The set (2.3.14) can be trivially derived from set (2.3.15).

We are now in a position to choose from the set s_{ij} an independent set of variables which will be equivalent to set ω , Δ , c , d , and v (or v') as given in equations (2.3.11). It turns out that s_{12} is a multiple of ω and since it is in terms of ω that we wish to investigate dispersion relations we choose s_{12} as one of our independent set. We obtain, after some algebra the following scheme:

$$\begin{aligned} S_{12} &= V & S_{13} &= C_{13} & S_{14} &= k_{14} - vV & S_{15} &= k_{15} - v'V \\ S_{23} &= k_{23} - V & S_{24} &= C_{24} & S_{25} &= C_{25} \\ S_{34} &= k_{34} + vV & S_{35} &= k_{35} + v'V \\ S_{45} &= k_{45} \end{aligned}$$

The set of independent variables which we have chosen are $s_{12} = V$, s_{13} , s_{24} , s_{25} and s_{34} and we have elected to fix the values of s_{13} , s_{24} and s_{25} at c_{13} , c_{24} and c_{25} : and we intend to fix v and v' subject to condition (2.3.4) but we reserve this right for the present. For the time being, we have essentially two variables s_{12} and s_{34} , or if we wish v and k_{34} . The other quantities are related to our independent set through equations (2.3.13) and (2.3.14) by:

$$k_{14} = M_{\pi}^2 + 2M_N^2 + c_{25} - c_{13} - k_{34}$$

$$k_{15} = 2M_{\pi}^2 + M_N^2 - c_{25} + k_{34}$$

$$k_{23} = 4M_{\pi}^2 + 2M_N^2 - c_{24} - c_{25}$$

(2.3.15)

$$k_{35} = M_N^2 - M_{\pi}^2 + c_{24} + c_{25} - c_{13} - k_{34}$$

$$k_{45} = 3M_{\pi}^2 + c_{13} - c_{24} - c_{25}$$

This, then, is our mode of description of the five point loop: we have assumed in equations (2.3.15) that $M_1 = M_3$ and $M_2 = M_4 = M_5$: Also, simply for convenience, we have labelled these masses M_N and M_{π} respectively.

(2.4) The Landau Curves

First of all we make some further definitions: we generalise our definition of the quantities p_{ij} as follows:

$$p_{ij} = p_{i,i+1} + p_{i+1,i+2} + \dots + p_{j-1,j}$$

where the arithmetic is again modulo 5, and the zero element is 5. This is clearly in agreement with the previous definition for $j = i + 1$.

Define also entities y_{ij} , μ_{ij} , λ_{ij} by the formulae:

$$y_{ij} = \frac{\mu_i^2 + \mu_j^2 - p_{ij}^2}{2\mu_i\mu_j} = \mu_{ij} - \lambda_{ij} p_{ij}^2, \quad (2.4.1)$$

$$y_{ii} = 1, \quad i < j : y_{ij} = y_{ji}, \quad i > j.$$

Both Tarski⁽⁸⁾, and Polkinghorne and Screaton⁽⁷⁾ use the fact that the analysis of single loop diagrams can be reduced to the study of quadratic forms to obtain the Landau equations in determinantal form, by quoting a standard theorem of algebra. In the case at present under discussion, this method can be employed to cast the equation of the leading Landau curve into the form:

$$\det y_{ij} = 0 \quad (2.4.2)$$

The lower order Landau curves are given by the vanishing of the various principal minors of the determinant in equation (2.4.2).

Writing equation (2.4.2) out in full in terms of our variables, as set out in Section (2.3), we obtain for the equation of the leading Landau curve:

$$\begin{vmatrix} 1 & \alpha_{12} & \mu_{13} - \lambda_{13}V & \beta & \alpha_{15} \\ \alpha_{12} & 1 & \alpha_{23} & \gamma + \lambda_{24}V & \delta - \lambda_{25}U \\ \mu_{13} - \lambda_{13}V & \alpha_{23} & 1 & \alpha_{34} & \mu_{35} - \lambda_{35}U - \lambda_{35}V \\ \beta & \gamma + \lambda_{24}V & \alpha_{34} & 1 & \alpha_{45} \\ \alpha_{15} & \delta - \lambda_{25}U & \mu_{35} - \lambda_{35}U - \lambda_{35}V & \alpha_{45} & 1 \end{vmatrix} = 0 \quad (2.4.3)$$

where

$$U = k_{34} - v'V$$

$$\alpha_{12} = \mu_{12} - \lambda_{12} M_N^2$$

$$\alpha_{15} = \mu_{15} - \lambda_{15} M_\pi^2$$

$$\alpha_{23} = \mu_{23} - \lambda_{23} M_\pi^2$$

$$\alpha_{34} = \mu_{34} - \lambda_{34} M_N^2$$

$$\alpha_{45} = \mu_{45} - \lambda_{45} M_\pi^2$$

$$\beta = \mu_{14} - \lambda_{14} c_{25} \quad (2.4.4)$$

$$\gamma = \mu_{24} - \lambda_{24} (4M_\pi^2 + 2M_N^2) + \lambda_{24} (c_{24} + c_{25})$$

$$\delta = \mu_{25} - \lambda_{25} (2M_\pi^2 + M_N^2) + \lambda_{25} c_{25}$$

Henceforth we will discuss matters in terms of u and v -- v is the incoming pion energy essentially and is the centre of our interest in the problem, but U , in itself, has no direct physical meaning although $u + v$ is an energy variable. It can be seen from equations (2.4.4) that to fix k_{34} is equivalent to fixing some linear combination of u and v -- the particular combination will depend on our choice of parameters λ and λ' .

(2.5) Normal Threshold Singularities

The lowest order singular curves are called the normal thresholds and they correspond to the vanishing of 2×2 principal minors of the determinant (2.4.3). Singularities of this type arise from the so-called contractions of the five-point loop, where three of the five internal lines have been shrunk to zero as illustrated in Fig. 27.

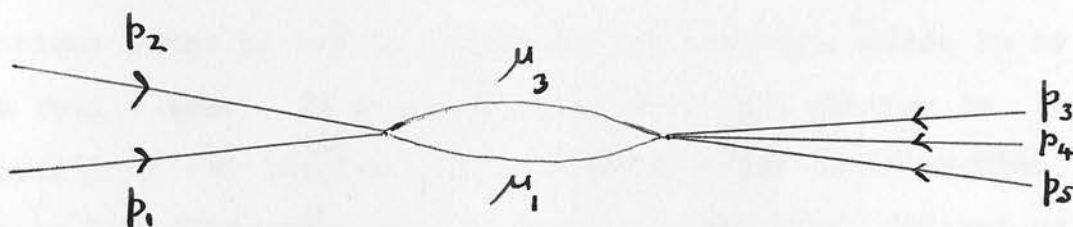


Fig. 27

It is evident that such graphs depend on only one of the variables s_{ij} (in the example of Fig. 27 the dependence is on s_{12} only). Unitarity (2.2.1) demands non-analytic behaviour at the point where the variable s_{ij} first reaches a value equal to the square of the sum of the masses in the two particle intermediate state (in the example of Fig. 27 this would be at $s_{12} = (\mu_1 + \mu_3)^2$, and we would have a normal threshold branch cut $s_{12} \geq (\mu_1 + \mu_3)^2$). Thus then it is clear that the expression for the total amplitude will include a collection of dispersion integrals of the type
$$\int \frac{\rho(s'_{ij}) ds'_{ij}}{s'_{ij} - s_{ij} - i\epsilon}$$

where the integration is over the cut region (in the example of Fig. 27 the range of integration would be $(\mu_1 + \mu_3)^2 \leq s'_{ij} \leq \infty$). Further, the structure of the variable scheme set up in Section (2.3) shows that, regardless of our choice of v and v' , there are s_{ij} which are linearly related both to positive and to negative values of v . Consequently there will arise two types of integral: $\int_{-\infty}^{\infty} \frac{g(v') dv'}{v' - v - i\varepsilon}$ and $\int_{-\infty}^{\infty} \frac{g(v') dv'}{v' - v + i\varepsilon}$. The significant difference between them is that they specify two distinct modes of taking limits in the complex v plane on to the real v axis. In cases where we have cuts running in opposite directions but without overlap there is no problem, but in the case where cuts do overlap a function, defined as a sum of two such terms as the above integrals, does not have the property of being the boundary value of an analytic function: in the region of overlap the two terms necessitate different prescriptions for limits on to the real axis. This type of difficulty was first pointed out by Landshoff and Trieman⁽⁴⁾. In our problem the lowest order Landau curves are, from (2.4.3), given by the following line pairs in the real (u, v) plane:

$$(\mu_{13} - \lambda_{13} v)^2 = 1$$

$$(\delta + \lambda_{24} v)^2 = 1 \quad (2.5.1)$$

$$(\delta - \lambda_{25} u)^2 = 1$$

$$(\mu_{35} - \lambda_{35} u - \lambda_{35} v)^2 = 1$$

Standard analysis^{*} reveals that not all these curves correspond to singularities on the physical sheet, and that those which do, lead to the branch lines given by the formulae:

$$\begin{aligned} \infty &\geq v \geq (\mu_1 + \mu_3)^2 \\ -\infty &\leq v \leq -2\mu_2\mu_4(1+\delta) \\ \infty &\geq u \geq 2\mu_2\mu_5(1+\delta) \\ \infty &\geq u+v \geq 2\mu_3\mu_5(\mu_3+1) = (\mu_3+\mu_5)^2 \end{aligned} \quad (2.5.2)$$

We call these respectively c_1 , c_2 , c_3 and c_4 and depict them in Fig. 28.

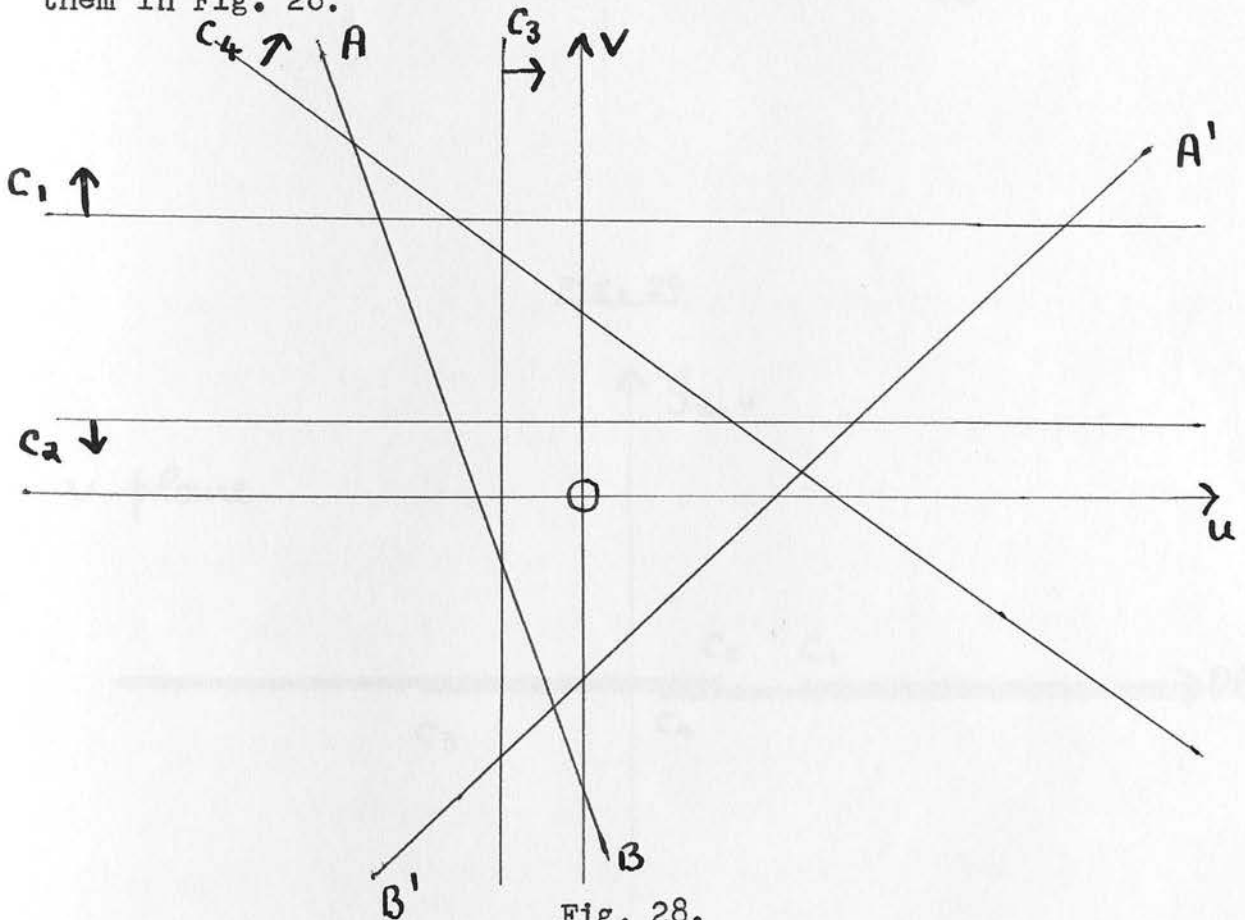


Fig. 28.

* The type of argument is detailed in the appendix.

Fixing k_{34} corresponds to drawing some arbitrary line (depending on choice of v or v') in the real (u, v) plane: then the values of v which correspond to the branch points are the intersections of this line with the c_i -- the directions in which the cuts run are given by the arrows in Fig. 28. For example, the sections by lines AB and A'B' are drawn in Figs. 29 and 30 respectively.

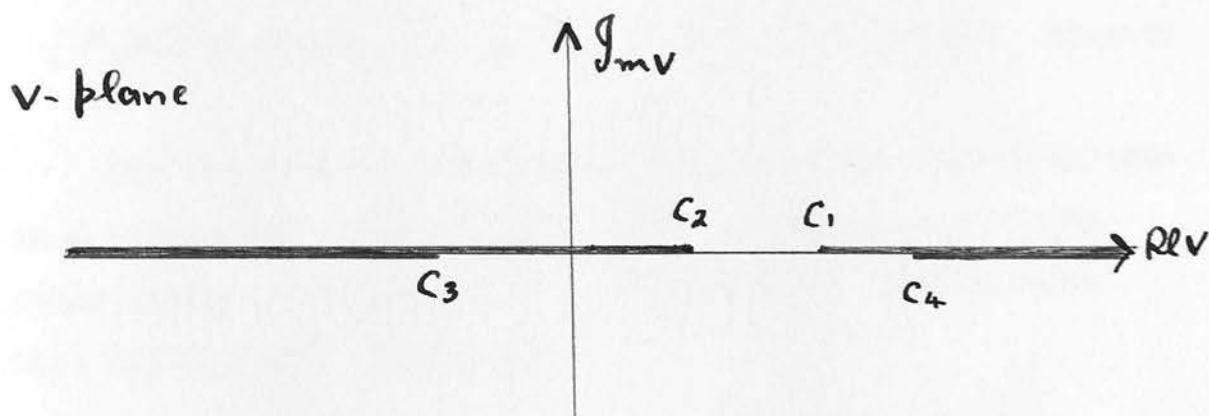


Fig. 29

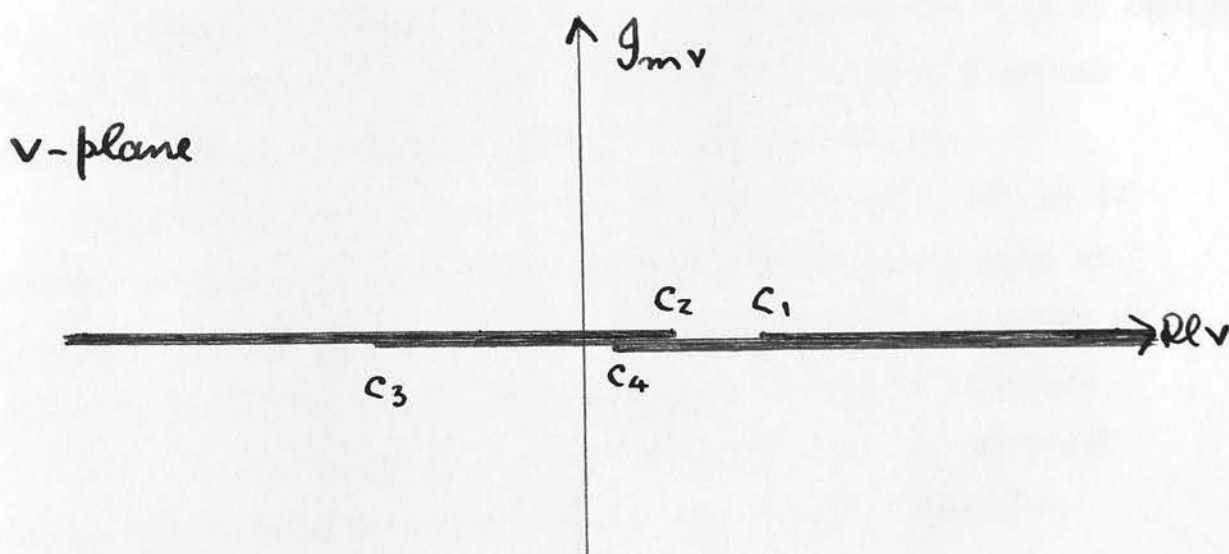


Fig. 30

For there to be any possibility of the cuts being non-overlapping in our problem c_1 must lie above c_2 . That is

$$(\mu_1 + \mu_3)^2 \geq -2\mu_2\mu_4 - 2\mu_2^2\mu_4\gamma \quad (2.5.3)$$

which reduces, using equations (2.4.1) and (2.4.4), to

$$(\mu_1 + \mu_3)^2 + (\mu_2 + \mu_4)^2 \geq 4M_\pi^2 + 2M_N^2 - (c_{24} + c_{25}) \quad (2.5.4)$$

For the sake of definiteness, we impose the selection rule that baryon number is conserved, and we set the mass of the other intermediate particles at the value of the pion mass. Then expression (2.5.4) becomes

$$c_{24} + c_{25} \geq 2M_\pi(M_\pi - 2M_N) \quad (2.5.5)$$

Physically we can always fix the momentum transfers c_{24} and c_{25} at negative values which satisfy the inequality (2.5.5) because $M_N \gg M_\pi$. Thus it will always be possible to find physical processes which avoid the overlap of cuts c_1 and c_2 .

The matter of the other cuts is not so simple, but it is fairly evident that for suitable choice of our parameters at physical values, overlap can be avoided for them also. Let us refer back to Fig. 28 and, for example, fix k_{34} in the case $U = 0$. In this instance we are fixing $u + v$ -- physical values of this quantity must exceed $(\mu_3 + \mu_5)^2$. Now, the

overlap of the c_1 and c_3 cuts can be avoided provided the intersection of c_1 with c_3 lies above the line c_4 , and provided also that we fix $u + v$ at a value less than its value at this intersection. The condition for avoiding overlapping cuts in this special case is:

$$(\mu_1 + \mu_3)^2 + 2\mu_2\mu_5(1+\delta) \geq (\mu_3 + \mu_5)^2 \quad (2.5.6)$$

This condition, using (2.4.1) and (2.4.4), reduces to

$$(\mu_1 + \mu_3)^2 + (\mu_2 + \mu_5)^2 - (\mu_3 + \mu_5)^2 \geq 2M_\pi^2 + M_N^2 - C_{25} \quad (2.5.7)$$

Subject to the same mass restrictions as we imposed above, this inequality becomes

$$C_{25} \geq M_\pi (M_\pi - 2M_N) \quad (2.5.8)$$

There clearly exist values of the momentum transfer c_{25} which are physical and which satisfy this latter inequality because $M_N \gg M_\pi$.

Our conclusion is thus not altogether satisfactory. It is that, while it is easy to construct physical examples which do not encounter the overlapping-cut problem, there certainly exist physical processes which do lead to such difficulties. Roughly speaking the difficult processes appear to be those which involve

large negative momentum transfers: one is, however, unworried in the case of small momentum transfers by normal threshold type singularities. The precise criteria for avoiding the difficulties are, from inequalities (2.5.4) and (2.5.7), for physical values of the s_{ij} :

$$C_{24} + C_{25} \geq 4M_{\pi}^2 + 2M_N^2 - (\mu_1 + \mu_3)^2 - (\mu_2 + \mu_4)^2 \quad (2.5.9)$$

$$C_{25} \geq 2M_{\pi}^2 + M_N^2 - (\mu_1 + \mu_3)^2 - (\mu_2 + \mu_5)^2 + (\mu_3 + \mu_5)^2 \quad (2.5.10)$$

Assuming, then, that (2.5.9) holds, we proceed to the investigation of higher order singularities. Notice that (2.5.9) is a necessary condition, while (2.5.10) refers only to a particular section of the diagram 28.

(2.6) Vertex Singularities

This is the most important section in this dissertation: in the author's opinion, the conclusions of this section are of such far reaching consequences, that upon them hang the future of the general applicability of dispersion techniques.

The vertex graph is one with three external vertices, and such graphs can be obtained from the five-point loop by shrinking to zero two of the internal lines as depicted in Fig. 31. These correspond to the vanishing of 3×3 minors of the determinant (2.4.2). Among the curves of this type are

ellipses which touch the normal thresholds and are singular, on the physical sheet, only on the minor arc between the tangency points as indicated in Fig. 32.* A line parallel to

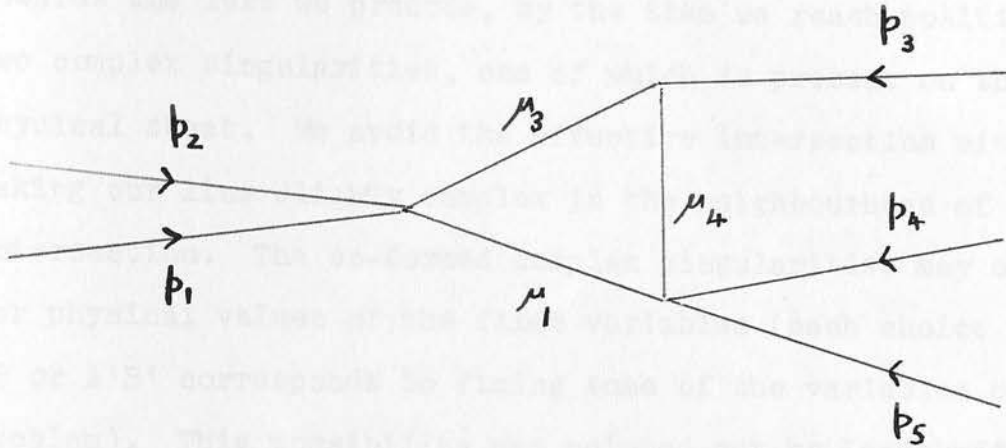


Fig. 31.

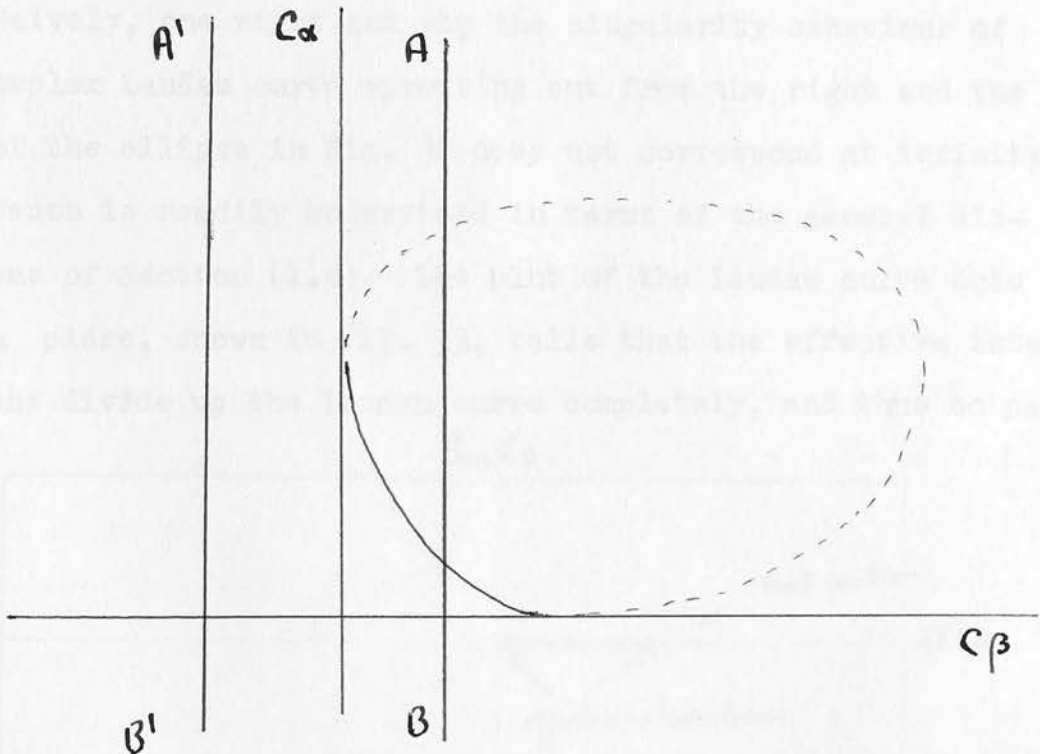


Fig. 32.

* Proof of this statement is implicit in the work of the appendix.

C_α has on it two points of possible singularity -- line AB defines two such points, one of which is singular on the physical sheet, and one which is not. If we move the line AB towards the left we produce, by the time we reach position A'B', two complex singularities, one of which is present on the physical sheet. We avoid the effective intersection with C_α by making our line slightly complex in the neighbourhood of this intersection. The so-formed complex singularities may occur for physical values of the fixed variables (each choice of line AB or A'B' corresponds to fixing some of the variables of the problem). This possibility was pointed out by Landshoff and Trieman, and it is this problem which we, in this section, must discuss in relation to our choice of variables for the five-point function.

Naively, one might ask why the singularity behaviour of the complex Landau curve sprouting out from the right and the left of the ellipse in Fig. 32 does not correspond at infinity. The reason is readily understood in terms of the general discussions of Section (1.4). The plot of the Landau curve onto the C_β plane, shown in Fig. 33, tells that the effective intersections divide up the Landau curve completely, and thus to pass

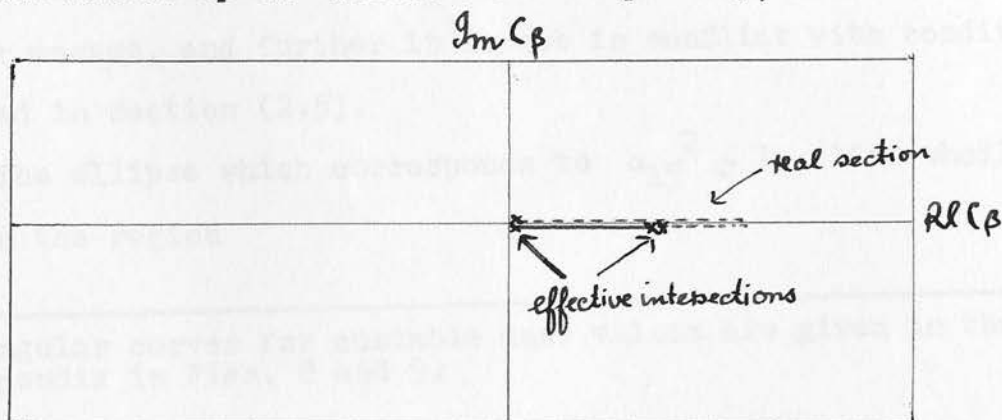


Fig. 33.

from the right hand portion to the left hand portion, while still remaining on the physical sheet, it is necessary actually to cross a branch cut.

The principal minors of $\det y_{ij}$, as can be seen from equation (2.4.3), are quadratic forms of two types, namely those which depend on two variables (u and v), and those which depend on only one variable (some linear combination of u and v). We discuss the two variable (type 1) system of equations first. There are three of them, and all three are conic sections. In an attempt to classify these conics it is immediately apparent that they will be ellipses if, and only if, $\alpha_{15}^2 \leq 1$, $\alpha_{23}^2 \leq 1$ and $\alpha_{45}^2 \leq 1$: these conditions can be rewritten in terms of particle masses and we obtain

$$\begin{aligned} (\mu_1 - \mu_5)^2 &\leq M_\pi^2 \leq (\mu_1 + \mu_5)^2 \\ (\mu_2 - \mu_3)^2 &\leq M_\pi^2 \leq (\mu_2 + \mu_3)^2 \\ (\mu_4 - \mu_5)^2 &\leq M_\pi^2 \leq (\mu_4 + \mu_5)^2 \end{aligned} \quad (2.6.1)$$

These are precisely the conditions of stability for the vertices of the diagram which involve a single external pion line. In normal circumstances we would impose such a condition on our masses, and further it is not in conflict with conditions imposed in Section (2.5).

The ellipse which corresponds to $\alpha_{15}^2 \leq 1$ lies wholly within the region

* Singular curves for unstable mass values are given in the appendix in Figs. 8 and 9.

$$(\mu_1 - \mu_3)^2 \leq v \leq (\mu_1 + \mu_3)^2$$

(2.6.2)

$$(\mu_3 - \mu_5)^2 \leq u+v \leq (\mu_3 + \mu_5)^2$$

and, in fact, it touches those lines which define the boundary of the region (2.6.2). Fig. 34 depicts this curve. Those Landau curves which are actually singular on the physical sheet are

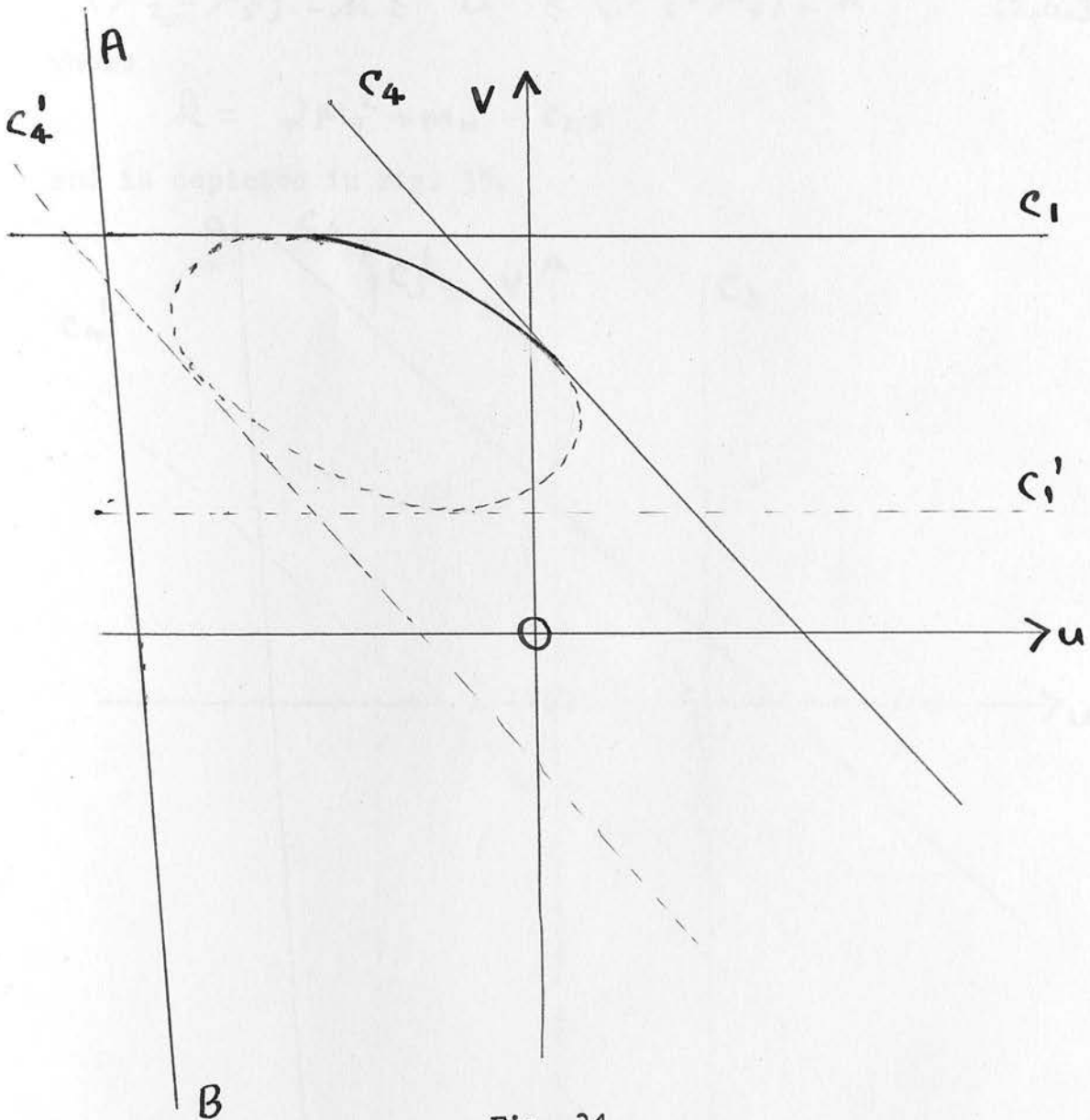


Fig. 34

drawn as solid lines (the method of discriminating between regions of singularity and non-singularity appears in the appendix: equation (16) onwards).

In a similar fashion the ellipse corresponding to $a_{23}^2 \leq 1$ lies inside the region

$$\begin{aligned} (\mu_3 - \mu_5)^2 &\leq u + v \leq (\mu_3 + \mu_5)^2 \\ (\mu_2 - \mu_5)^2 - k &\leq u \leq (\mu_2 + \mu_5)^2 - k \end{aligned} \quad (2.6.3)$$

where

$$k = 2M_\pi^2 + M_N^2 - c_{25}$$

and is depicted in Fig. 35.

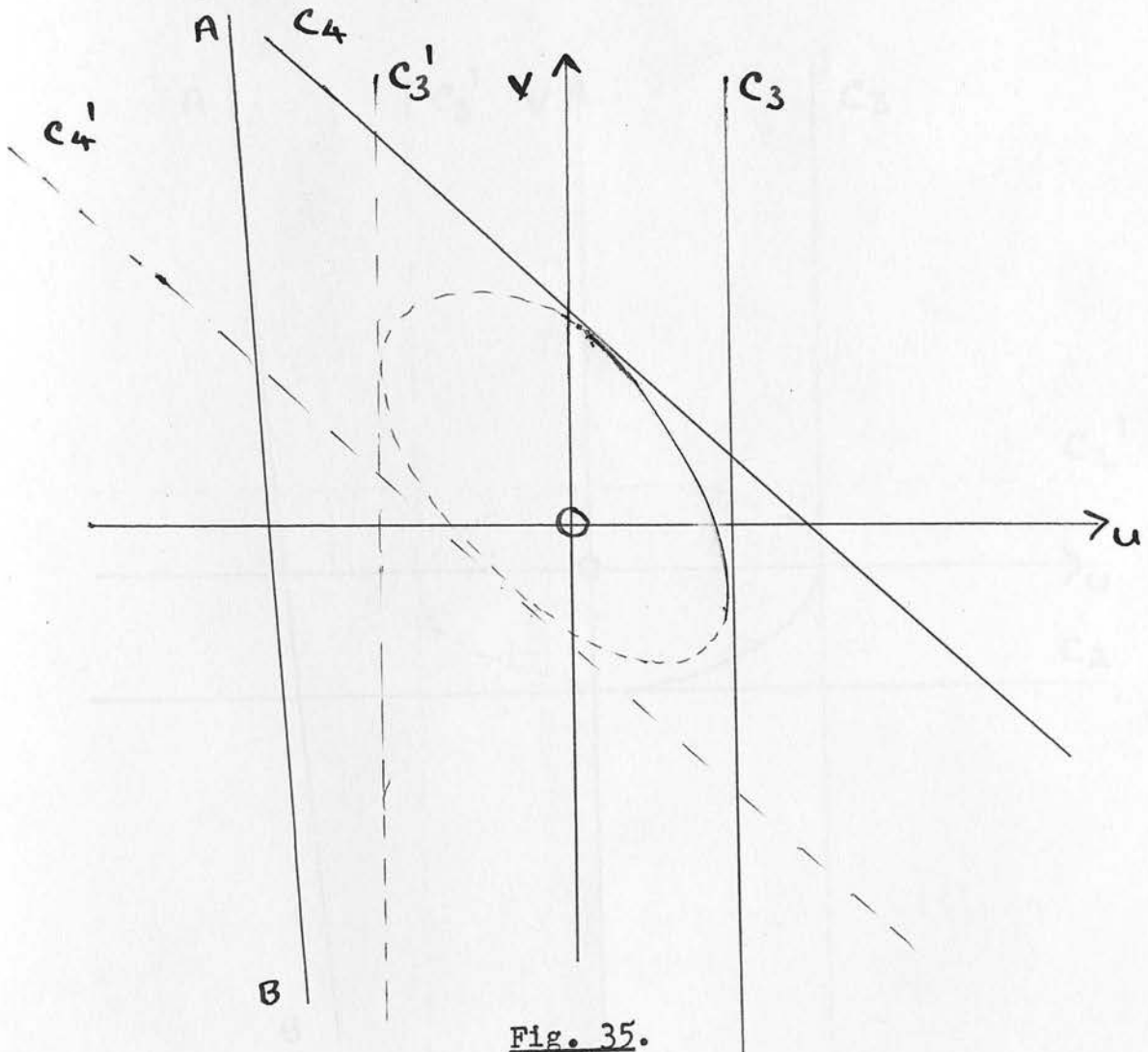


Fig. 35.

Finally the ellipse corresponding to $a_{45}^2 \leq 1$ lies within the region:

$$\begin{aligned} (\mu_2 - \mu_5)^2 - k &\leq u \leq (\mu_2 + \mu_5)^2 - k \\ -(\mu_2 + \mu_4)^2 + k' &\leq v \leq -(\mu_2 - \mu_4)^2 + k' \end{aligned} \quad (2.6.4)$$

where

$$k' = 4M_0^2 + 2M_N^2 - c_{24} - c_{25}$$

and is sketched in Fig. 36.

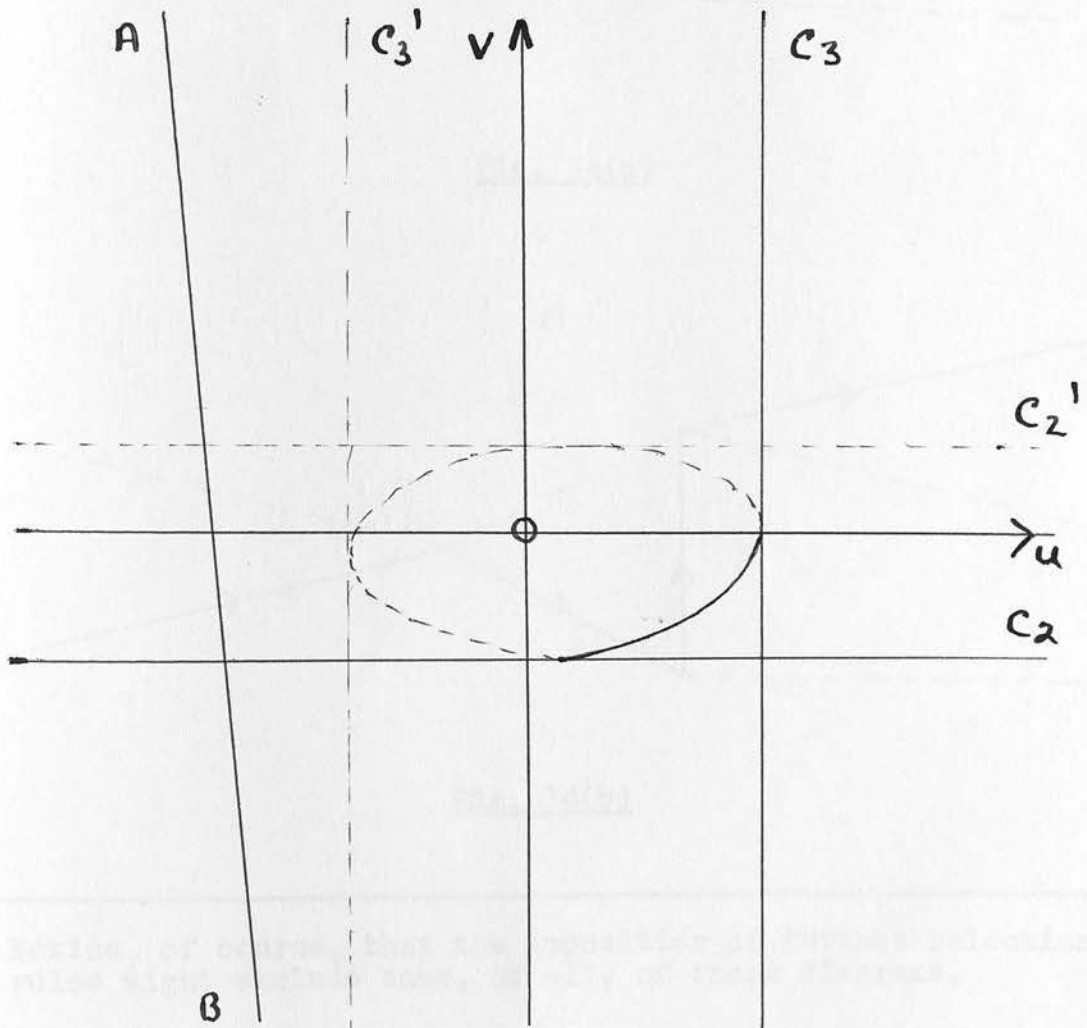


Fig. 36.

In the case where baryon number is conserved the Landau curves of Figs. 34, 35 and 36, correspond to the processes depicted below.* The numbering of the figures indicates to which Landau curve they belong: the symbols (a) and (b) refer to two possible ways of conserving baryon number in the original five-point loop diagram.

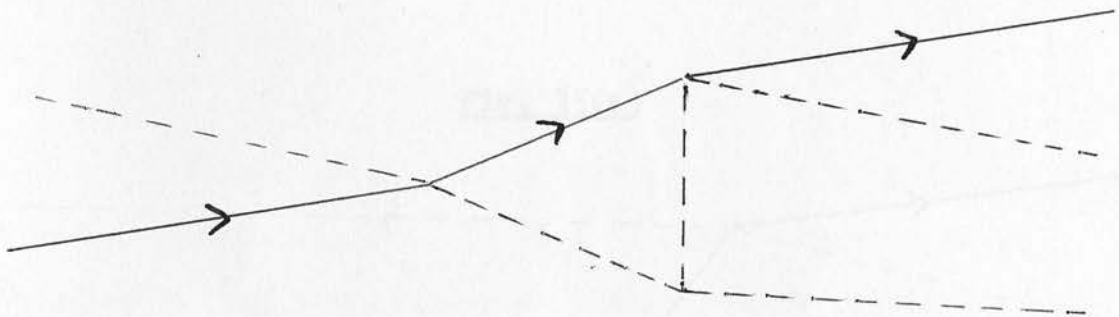


Fig. 34(a)

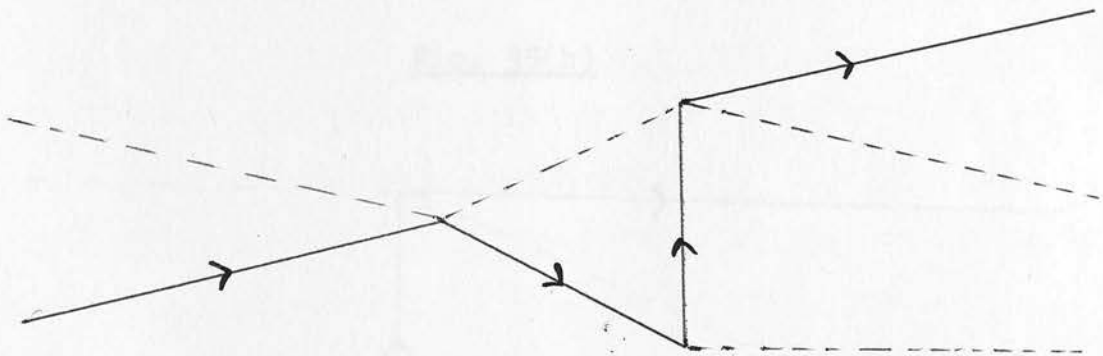


Fig. 34(b)

* Notice, of course, that the imposition of further selection rules might exclude some, or all, of these diagrams.

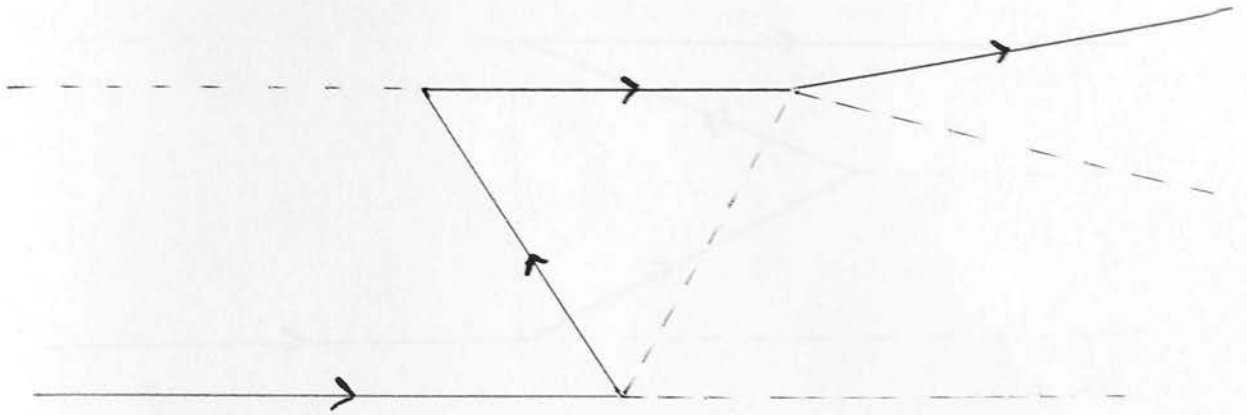


Fig. 35(a)

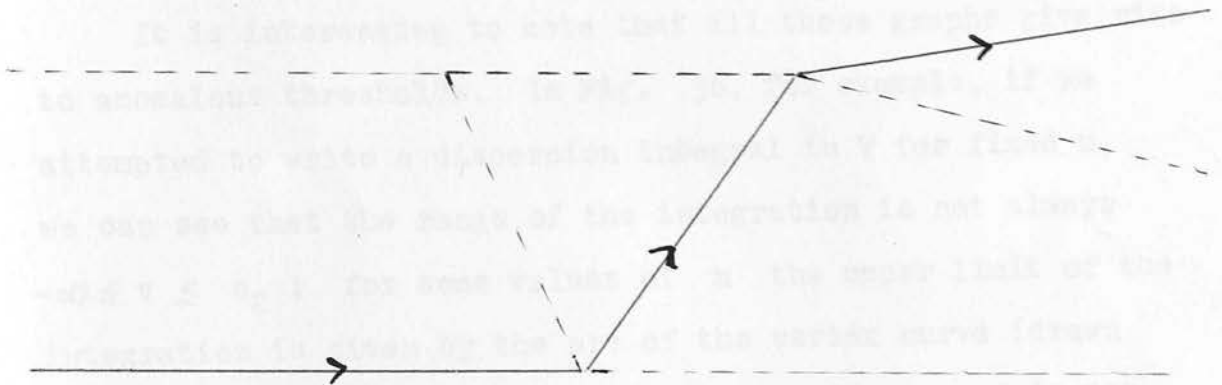


Fig. 35(b)

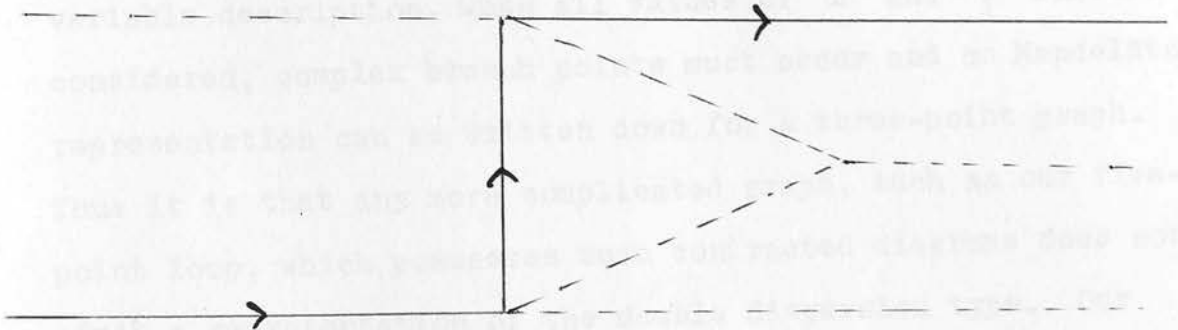


Fig. 36(a)

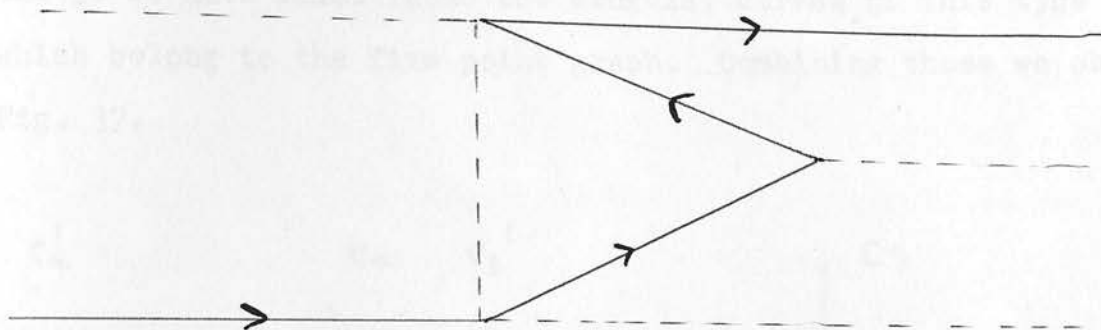


Fig. 36(b)

It is interesting to note that all these graphs give rise to anomalous thresholds. In Fig. 36, for example, if we attempted to write a dispersion integral in V for fixed u , we can see that the range of the integration is not always $-\infty \leq v \leq c_2$: for some values of u the upper limit of the integration is given by the arc of the vertex curve (drawn solid in Figs.). Such singularities were discovered in perturbation theory and do not correspond to thresholds of the normal type: they are called anomalous thresholds. Their existence leads to complex singular points so that, in a two variable description, when all values of u and v must be considered, complex branch points must occur and no Mandelstam representation can be written down for a three-point graph. Thus it is that any more complicated graph, such as our five-point loop, which possesses such contracted diagrams does not admit a representation of the double dispersion type. Our problem is to decide whether the vertex graphs also exclude the possibility of single-dispersion integrals. In Figs. 34, 35

and 36 we have illustrated the singular curves of this type which belong to the five point graph. Combining these we obtain Fig. 37.

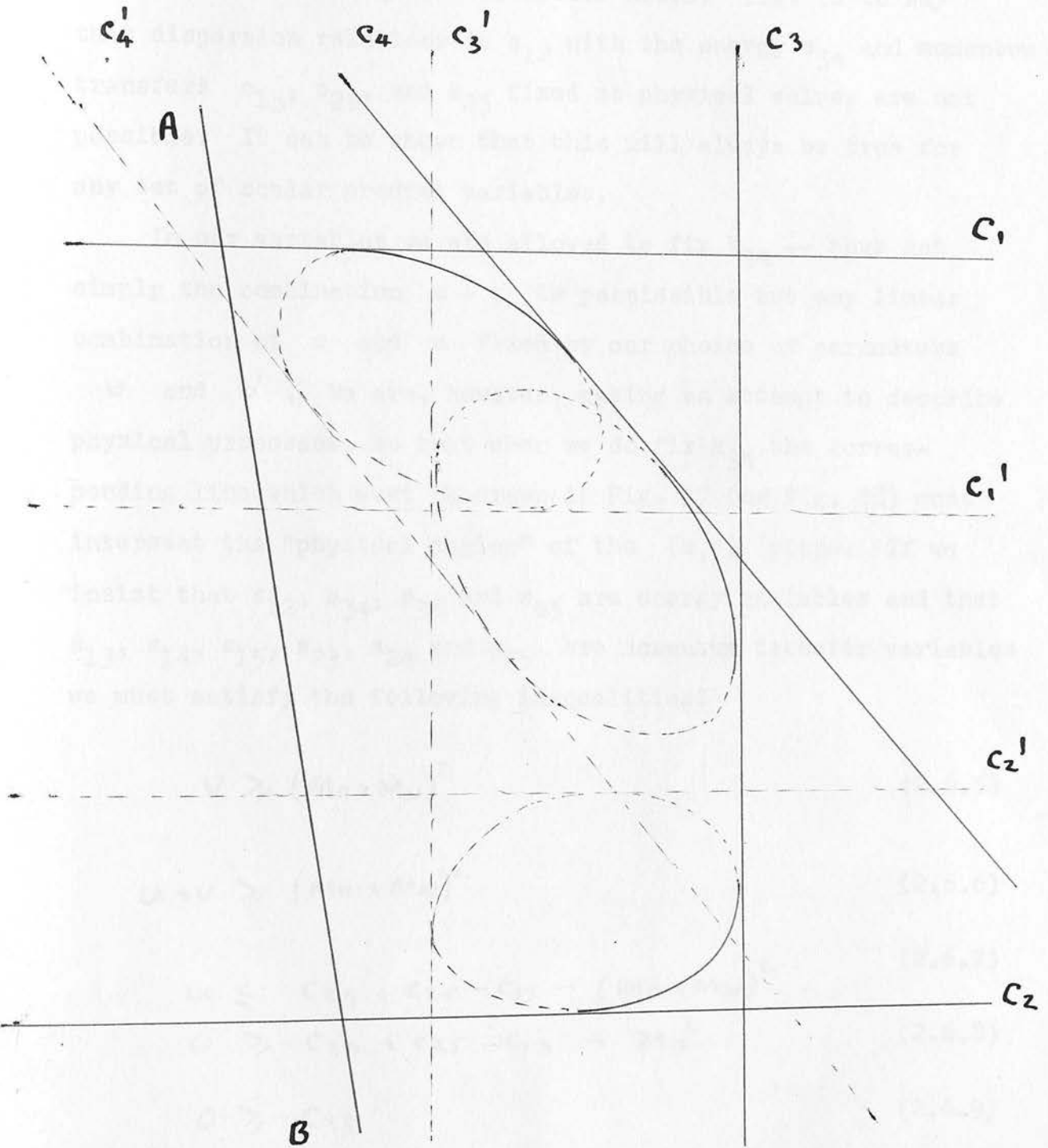


Fig. 37.

It is immediately apparent that if we fix $u + v$ at a physical value, i.e. $u + v$ above the value given by c_4 , there will always arise complex branch points in the physical sheet according to the mechanism described above. That is to say that dispersion relations in s_{12} with the energy s_{34} and momentum transfers c_{13} , c_{24} , and c_{25} fixed at physical values are not possible. It can be shown that this will always be true for any set of scalar product variables.

In our variables we are allowed to fix k_{34} -- thus not simply the combination $u + v$ is permissible but any linear combination of u and v fixed by our choice of parameters v and v' . We are, however, making an attempt to describe physical processes, so that when we do fix k_{34} the corresponding line which must be drawn in Fig. 37 (or Fig. 28) must intersect the "physical region" of the (u, v) plane. If we insist that s_{12} , s_{34} , s_{35} and s_{45} are energy variables and that s_{13} , s_{14} , s_{15} , s_{23} , s_{24} and s_{25} are momentum transfer variables we must satisfy the following inequalities:

$$v \geq (M_\pi + M_N)^2 \quad (2.6.5)$$

$$u + v \geq (M_\pi + M_N)^2 \quad (2.6.6)$$

$$u \leq c_{24} + c_{25} - c_{13} - (M_\pi + M_N)^2 \quad (2.6.7)$$

$$0 \geq c_{24} + c_{25} - c_{13} + M_\pi^2 \quad (2.6.8)$$

$$0 \geq c_{13} \quad (2.6.9)$$

$$u + v \geq c_{25} - c_{13} + M_N^2 + 2M_N M_\pi \quad (2.6.10)$$

$$u \leq c_{25} - M_{\pi}^2 - 2 M_N M_{\pi} \quad (2.6.11)$$

$$v \geq -c_{24} - c_{25} + M_N^2 + 3 M_{\pi}^2 + 2 M_N M_{\pi} \quad (2.6.12)$$

$$0 \geq c_{24} \quad (2.6.13)$$

$$0 \geq c_{25} \quad (2.6.14)$$

Inequalities (2.6.9) and (2.6.13), (2.6.14), force us to choose c_{13} , c_{24} , and c_{25} negative; inequalities (2.5.9) and (2.6.8) (the former being a necessary condition for avoiding overlapping cuts) restrict this choice somewhat -- we still have, however, considerable freedom of choice.

Inequalities (2.6.5) and (2.6.12) set a lower bound for v .

Inequalities (2.6.6) and (2.6.10) set a lower bound for $u + v$.

Inequalities (2.6.7) and (2.6.11) set an upper bound for u .

Thus, while the precise definition of the region is rather complicated we can assert that it is roughly that shaded in Fig. 38.

Thus we reach the conclusion that a section such as AB in Fig. 37 both intersects the "physical region" and avoids complex branch points on the physical sheet. Further, inspection of Fig. 28, tells us immediately that inequality (2.5.9) is not only a necessary condition for avoiding overlapping normal threshold cuts but, in this case, that it is

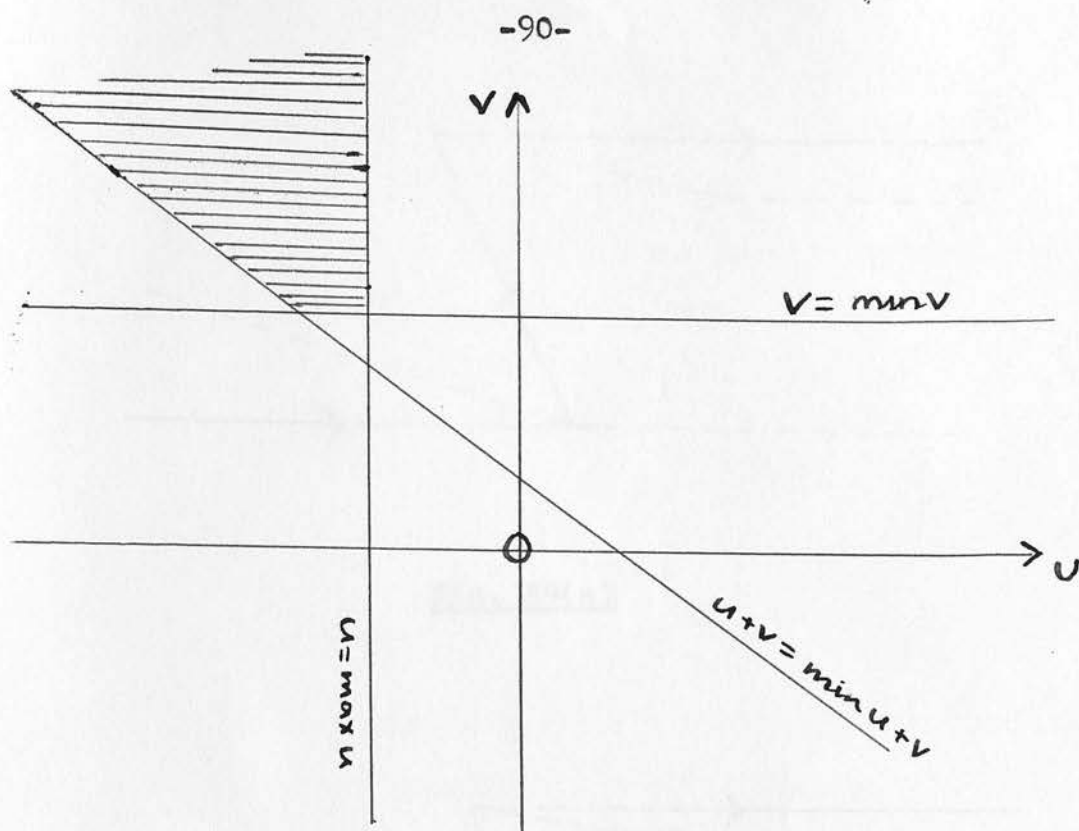


Fig. 38

also sufficient. We conclude, then, that values v and v' (subject to $v + v' = 1$) can be chosen in such a way that the graphs so far considered present no obstacles to the writing of single dispersion relations.

We turn now to the singular vertex curves of type 2 which depend on only one linear combination of variables u and v . Amongst those arises the graph obtained from the five-point loop by removing the lines 1 and 2 and which corresponds to a process of the type illustrated in Fig. 39. The quadratics, in such a case, represent line pairs: that corresponding to the process of Figs. 39, is given by the equation

$$1 - \alpha_{34}^2 - \alpha_{45}^2 - (\mu_{35} - \alpha_{35}u - \alpha_{35}v)^2$$

$$+ 2\alpha_{34}\alpha_{45}(\mu_{35} - \alpha_{35}u - \alpha_{35}v) = 0$$

(2.6.15)

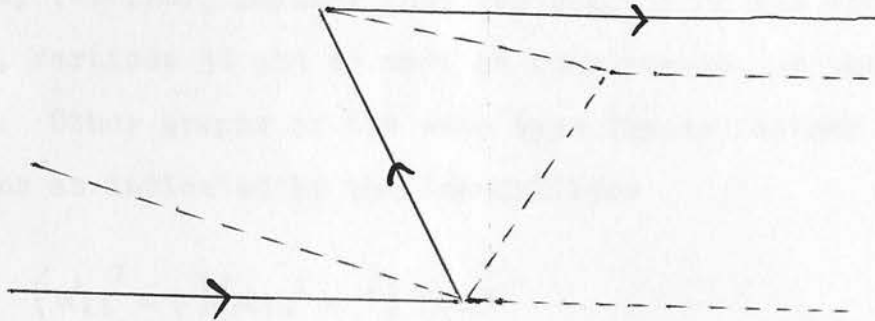


Fig. 39(a)

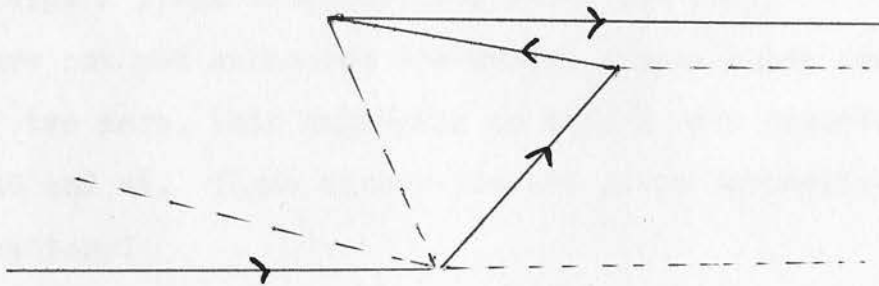


Fig. 39(b)

This equation will represent a real line pair if and only if the inequality

$$(\alpha_{34}^2 - 1)(\alpha_{45}^2 - 1) \geq 0 \quad (2.6.16)$$

is satisfied.

As before, an equation of the form $\alpha_{ij}^2 \leq 1$, is simply a stability condition on the masses at the vertex ij . Thus,

inequality (2.6.16), implies that for reality of the line pair (2.6.15), vertices 34 and 45 must be both stable, or both unstable. Other graphs of the same type impose further such conditions as indicated by the inequalities

$$\begin{aligned}(\alpha_{23}^2 - 1)(\alpha_{34}^2 - 1) &\geq 0 \\(\alpha_{12}^2 - 1)(\alpha_{15}^2 - 1) &\geq 0 \\(\alpha_{12}^2 - 1)(\alpha_{23}^2 - 1) &\geq 0\end{aligned}\tag{2.6.17}$$

Thus, if we continue to deal with stable mass values, these curves of type 2 yield singularities which are real.

We have not yet exhausted the vertex graphs which are possible: two more, both belonging to type 2, are depicted in Figs. 40 and 41. These line pairs are given respectively by the equations:

$$\delta + \alpha_{24} v = \alpha_{12} \beta \pm i \sqrt{(\beta^2 - 1)(1 - \alpha_{12}^2)}\tag{2.6.18}$$

$$\mu_{13} - \alpha_{13} v = \alpha_{34} \beta \pm i \sqrt{(\beta^2 - 1)(1 - \alpha_{12}^2)}\tag{2.6.19}$$

If now the vertex 12 is stable, and the singular lines real we must have $\beta^2 \leq 1$. But

$$\begin{aligned}\frac{\beta^2 - 1}{\alpha_{14}^2} &= (\mu_1 - \mu_4)^2 (\mu_1 + \mu_4)^2 + c_{25}^2 \\&\quad - 2 \left[(\mu_1 - \mu_4)^2 + (\mu_1 + \mu_4)^2 \right] c_{25}\end{aligned}\tag{2.6.20}$$

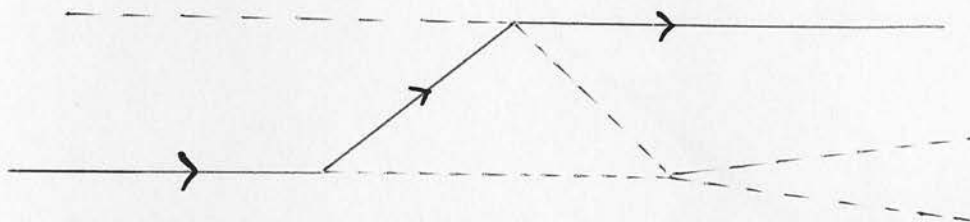


Fig. 40(a)

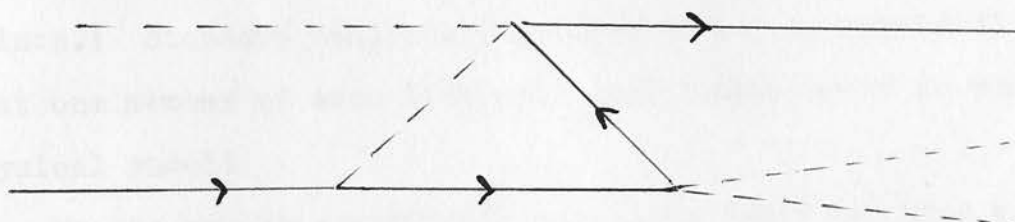


Fig. 40(b)

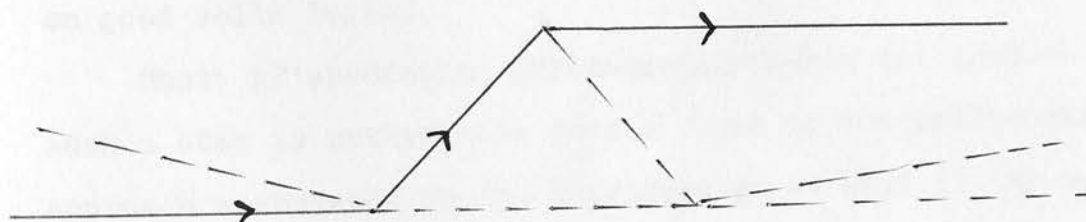


Fig. 41(a)

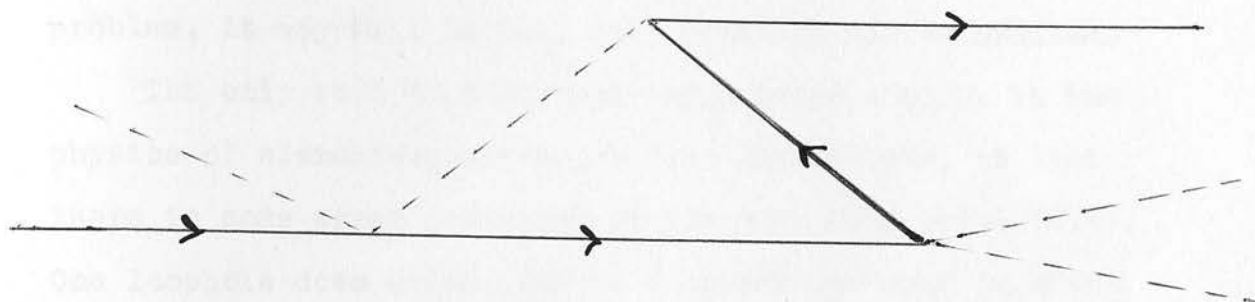


Fig. 41(b)

and the fact that $c_{25} \leq 0$, implies that $\beta^2 \gg 1$. Thus we have been forced into a situation where there exist complex branch points.!! Standard analysis (illustrated in the appendix) shows that one member of each line pair must always occur on the physical sheet!

We are now in a difficult position: what has gone wrong?

Perhaps complex singularities of this type are a fiction perpetrated on theoretical physics by perturbative techniques -- for perhaps, in the perturbation series as a whole, the complex singularities will somehow cancel when certain terms are added together -- but, these suggestions are based more on hope than on good solid logic.

Short of abandoning perturbation theory for good -- and such a step is unthinkable in the light of the philosophy of approach enunciated in the introduction -- what is the way out of the present impasse?

One does not relish the idea of coping with the complexities

of imaginary branch cuts in the case of diagrams of arbitrary complication: but, although the author does not share the confidence of Mandelstam^{*} that the matter is not a major problem, it may well be that the situation can be handled.

The only real hope of avoiding finding a niche in the physics of elementary particles for complex cuts, is that there is some error contained in the work of Section (2.6). One loophole does exist, and it concerns the mode in which "locality" (or the axiom of causality) has been built into perturbation theory: that one should look into this matter was suggested to the present author by Polkinghorne.⁽⁶⁾

(2.7) Locality

Let us agree to omit multiplicative numerical factors. Then the production process $p + k \rightarrow p' + k' + k''$ can be represented in terms of Heisenberg field operators as:

$$_{out} \langle p', k', k'' | p, k \rangle_{in} -_{in} \langle p', k', k'' | p, k \rangle_{in} = \int d^4x_1 \dots d^4x_5$$

$$e^{i \left[p'x_1 + k'x_2 + k''x_3 - px_4 - kx_5 \right]} \langle 0 | T(j_1 j_2 j_3 j_4 j_5) | 0 \rangle \quad (2.7.1)$$

The transition from Heisenberg operators to an interaction representation (these two coinciding at $t = \tau$) is effected by a replacement $j(x) \rightarrow U^\dagger(t, \tau) j(x) U(t, \tau)$ and $|0\rangle \rightarrow U(\tau, -\infty) |0\rangle$. One obtains

* See Section (1.5), page 49.

$$\langle 0 | T \prod_{k=1}^5 J_k | 0 \rangle = \langle 0 | U(\infty, -\infty) \prod_{k=1}^5 J_k | 0 \rangle \quad (2.7.2)$$

where

$$U(\infty, -\infty) = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int_{-\infty}^{+\infty} d^4 \gamma_1 \dots d^4 \gamma_n$$

$$T \{ H_I(\gamma_1) \dots H_I(\gamma_n) \} = 1 + \dots \quad (2.7.3)$$

H_I is the interaction Hamiltonian and for a trilinear interaction, e.g. vertices of the type shown in Fig. 42, it will be of the form $H_I \sim \bar{\Psi} \delta \Psi \phi$ with $j \sim \bar{\Psi} \delta \Psi$, where

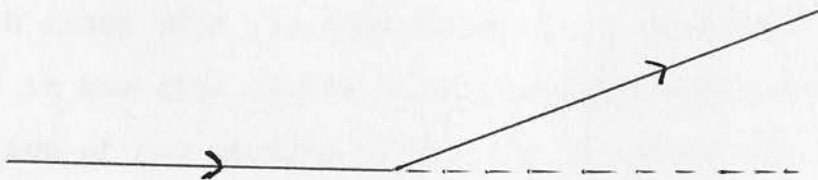


Fig. 42

ψ is a fermion and ϕ a boson field. The diagram with a single loop is obtained by taking only the lowest order term in the coupling strength in equation (2.7.3), i.e. $U = 1$. Our model has been of scalar fields only, and the general single loop diagram with scalar particles of mass μ can be artificially constructed by setting $J_k = N \phi_{k-1} \phi_k$, $k = 1, 2, \dots \pmod{5}$.^{*}

* N denotes Normal product.

An application of Wick's theorem gives in a straightforward manner for the Fourier transform of the T product

$$\int \frac{d^4 k_1 \dots d^4 k_5 \delta(p+k-p'-k'-k'')}{(k_1^2 - \mu_1^2 + i\varepsilon) \dots (k_5^2 - \mu_5^2 + i\varepsilon)} \times \quad (2.7.4)$$

$$\times \delta(p+k_1-k_2) \delta(k_1+k_2-k_3) \delta(-p'+k_3-k_4) \delta(-k'+k_4-k_5)$$

Now in field theory the equation (2.7.1) can be written in terms of a retarded product (R) rather than a time-ordered product (T). This involves changing the sign of the imaginary parts of the denominators in a well defined way. The question thus arises whether or not the definition of the Feynman function is thereby altered -- in the physical scattering regions the R and T product formulations are identical but what is important, for our purposes, is whether we are dealing in both cases with the same sheet of the function $F(z)$. A change in the sign of the imaginary parts may alter the prescription of how we must thread our way round the branch points to reach the physical sheet.

In the case relevant to the singularities of Section (2.6) it is shown in the appendix that there always exist complex singularities on the physical sheet regardless of the way in which the signs of the imaginary parts in the denominators have been prescribed. Furthermore, it is clear from equations (2.6.18) and (2.6.19) that on the physical sheet there will exist two complex branch points in the variable v -- one in the upper, and one in the lower, half-plane -- so that we cannot even prove half-plane analyticity of the five-point

function because of the vertex contractions.

(2.8) Conclusions

In the work described in this chapter we have run into difficulties without ever considering the leading Landau curve for the five-point loop, or even the curves of second highest order. Unless these difficulties can be overcome there is little point in proceeding further. Let us take stock of the situation.

The Mandelstam conjecture for elastic processes suggested that the collision amplitude was the boundary value of a function regular apart from poles in a space with suitable real branch cuts. Although considerable doubt has been cast on the truth of this conjecture (see Chapter 1), it has, for practical purposes, proved to be a good approximation in a variety of problems. However, it is beginning to be apparent that complex singularities do play some role -- how important a role it is not possible to say.

The analogue of the Mandelstam conjecture for production processes is some sort of analyticity in the space of the five invariants which describe the problem. This is rather a formidable matter, and, as a first step the present author was content to discuss analyticity in one variable only. The hope cherished was that the collision amplitude would be analytic, in some plane with suitable real branch cuts. However, almost

at the first stage, we find that the branch cuts are certainly not all real. Scalar product variables always lead to complex singularities, as do the variables used by the present author in his investigations.

The conclusion is inescapable. Complex singularities simply exist, and we must face the fact!

Having reached this conclusion, there is not a great deal of difficulty in principle in generalising the existing techniques. Complex contours of integration are dealt with in the same manner as real ones, and Cauchy's integral theorem applied to a suitable contour. The discontinuity across a branch cut is calculable in terms of the values of the amplitude on two adjacent sheets, and a generalisation of the unitarity relation can indubitably be employed. For any given diagram, we know that the complex singularities cannot divide up the space of the invariants, and so the isolation of regions of physical interest is not possible. There does exist a difficulty, however, when the perturbation series is summed, for then a complex jungle of branch cuts may divide the space up in an undesirable fashion.

At any rate, we can write down an integral representation for any given diagram, but the matter appears very complicated: it is conceivable that some pattern of singularities might emerge which would lead to simplifications, but certainly the future of dispersive techniques depends critically on dealing with complex singularities.

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APPENDIX

We shall now discuss, in some detail, how, starting from a given Feynman integral, and the Landau equations, one may obtain the Landau curves and how one decides whether or not a particular singularity appears on a given sheet of the function $F(z)$. The method of solution employed here is suggested by the general theorems of Section (2.1). It is interesting to notice that the author's method, unlike the conventional methods, yields all the solutions of the Landau equations: that is, not only the Landau singularities but also the so-called non-Landau singularities. When the present author first found extra solutions he did not investigate their significance. However, at the Symposium on Strong Interactions (Rutherford High Energy Laboratory, December 1961) it was brought to his attention that these extra solutions were the non-Landau singularities discovered by Cutkosky⁽¹⁾ and subsequently discussed by Fairlie, Landshoff, Nuttall and Polkinghorne.⁽²⁾

As an illustration, we discuss the three point single loop, which is necessary in connection with Section (2.7). In Fig. 1, we measure the external momenta p_i as ingoing, and denote the internal momenta by q_i . The Landau equations are:

$$\left. \begin{aligned} p_1 &= q_3 - q_2 \\ p_2 &= q_1 - q_3 \\ p_3 &= q_2 - q_1 \end{aligned} \right\} \quad (1)$$

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 0 \quad (2)$$

$$\left. \begin{aligned} p_1^2 &= z_1 \\ p_2^2 &= z_2 \\ p_3^2 &= z_3 \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} q_1^2 &= m_1^2 \\ q_2^2 &= m_2^2 \\ q_3^2 &= m_3^2 \end{aligned} \right\} \quad (4)$$

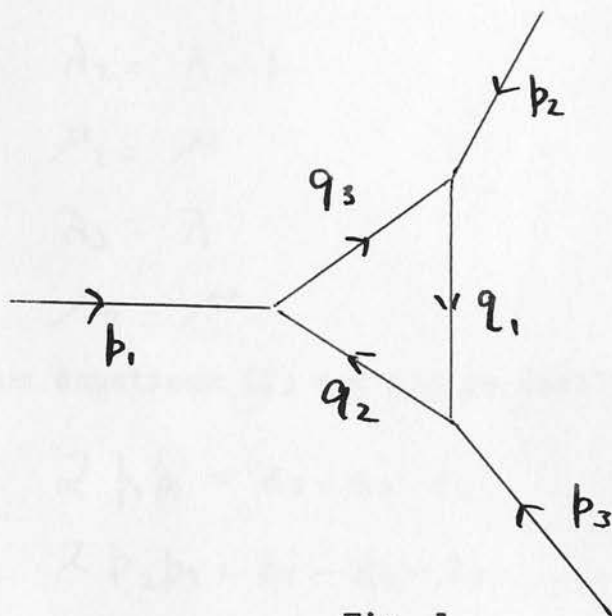


Fig. 1.

Because of overall energy-momentum conservation the set of equations is essentially two dimensional, and we can choose, as a basic set, p_1 and p_2 -- all other vectors must be expressible as linear combinations of these. Let us write

$$q_i = \lambda_i p_1 + \mu_i p_2 \quad i=1,2,3 \quad (5)$$

The equations (1) are now equivalent to the following set:

$$\left. \begin{aligned} 1 &= \lambda_2 - \lambda_3 \\ 0 &= \mu_3 - \mu_2 \\ 0 &= \lambda_1 - \lambda_3 \\ 1 &= \mu_1 - \mu_3 \end{aligned} \right\} \quad (6)$$

We solve these in terms of two arbitrary constants λ and μ as follows:

$$\begin{aligned}\lambda_1 &= \lambda \\ \mu_1 &= \mu + 1 \\ \lambda_2 &= \lambda - 1 \\ \mu_2 &= \mu \\ \lambda_3 &= \lambda \\ \mu_3 &= \mu\end{aligned}\tag{7}$$

From equations (1) and (3) we easily find that

$$\begin{aligned}2 p_1 p_2 &= z_3 - z_1 - z_2 \\ 2 p_2 p_3 &= z_1 - z_2 - z_3 \\ 2 p_3 p_1 &= z_2 - z_3 - z_1\end{aligned}\tag{8}$$

If, for convenience we take $m_1^2 = 1$, $i = 1, 2, 3$ equations (8) and (4) lead to:

$$\begin{aligned}\lambda^2 z_1 + \lambda \mu (z_3 - z_2 - z_1) + \mu^2 z_2 &= 1 \\ (\lambda - 1)^2 z_1 + (\lambda - 1) \mu (z_3 - z_2 - z_1) + \mu^2 z_2 &= 1 \\ \lambda^2 z_1 + \lambda (\mu + 1) (z_3 - z_2 - z_1) + (1 + \mu)^2 &= 1\end{aligned}\tag{9}$$

The eliminant of λ and μ from the set (9) is easily found: it is

$$\begin{aligned}\left\{ 2(z_1 z_2 + z_2 z_3 + z_3 z_1) - (z_1^2 + z_2^2 + z_3^2) \right\} \left\{ 2(z_1 z_2 + z_2 z_3 \right. \\ \left. + z_3 z_1) - (z_1^2 + z_2^2 + z_3^2) - z_1 z_2 z_3 \right\} = 0\end{aligned}\tag{10}$$

The second factor in equation (10) gives the usual Landau curve for the problem. The other factor actually arises from values of the p_i which fail to span a two-dimensional space: singularities belonging to this factor do not appear on the physical sheet of $F(z)$. They have been called, rather inappropriately, the non-Landau singularities.

To investigate whether or not, at a particular point on a Landau curve, $F(z)$ is singular, on some sheet, one must first obtain the set of points $\alpha(z)$ (see Section 1.1) at which the denominator vanishes: we must in fact express the Feynman denominator as a function of the α and the z variables. This is a matter of algebra. Rules for this purpose may be formulated: they are justified implicitly by work of Symanzik⁽³⁾: our version of these rules is as follows:

1) Remove from the graph under consideration as many lines as possible without disconnecting the vertices: call such a class T_i -- clearly there will be several such classes for any given diagram (the index i summarises the labels of the lines removed).

2) Multiply together the α 's belonging to lines in each class T_i and add together all the products so formed. Denote the result of this operation by C .^{*}

3) Select a set of invariants I_j which form independent variables for the problem.^{**}

* For a diagram for which the set T_i is always null define C as unity.

** We include the squares of the external momenta.

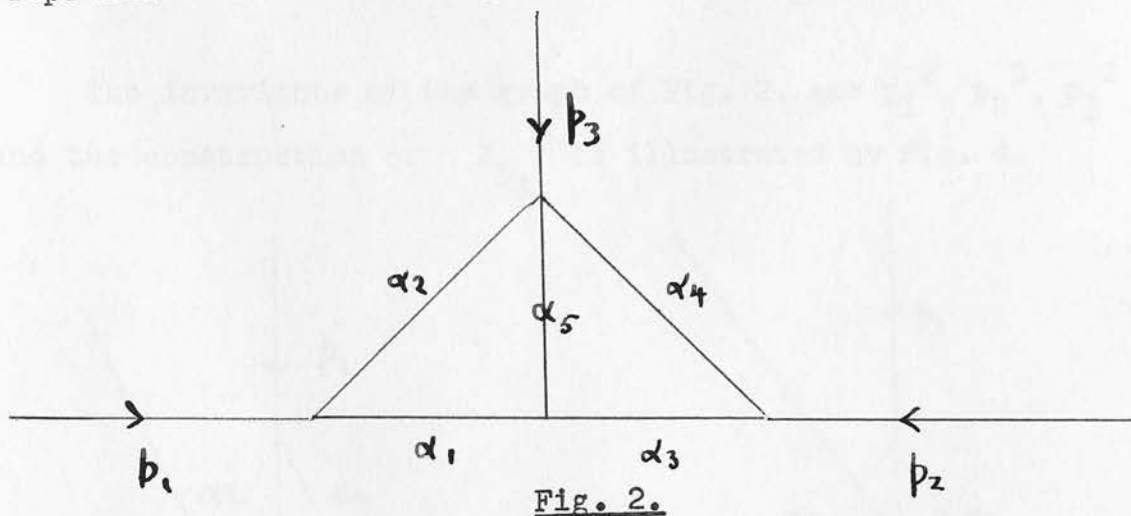
4) Slice the Feynman graph in such a way as to isolate the external momenta from which a particular invariant I_j is formed. Discounting lines of the graph actually sliced, the original diagram is split into two sub-diagrams. For each sub-graph construct the entity C according to the rules 1) and 2) -- call these CA and CB : now, multiply together the α 's belonging to lines actually sliced and $CA.CB$. Of course there will be several slices possible for each I_j : call the sum of all possible such products E_{I_j} : there will be as many E_{I_j} as there are independent variables.

5) The denominator function is now expressible as

$$D = \sum I_j E_{I_j} - C \sum \alpha \mu^2 \quad (11)$$

In the first sum we have one term for each invariant: in the second, the summation goes over all lines of the graph.

As an example consider the diagram of Fig. 2. The various ways of removing lines are drawn in Fig. 3 where crosses represent vertices. The quantity C is just the sum of



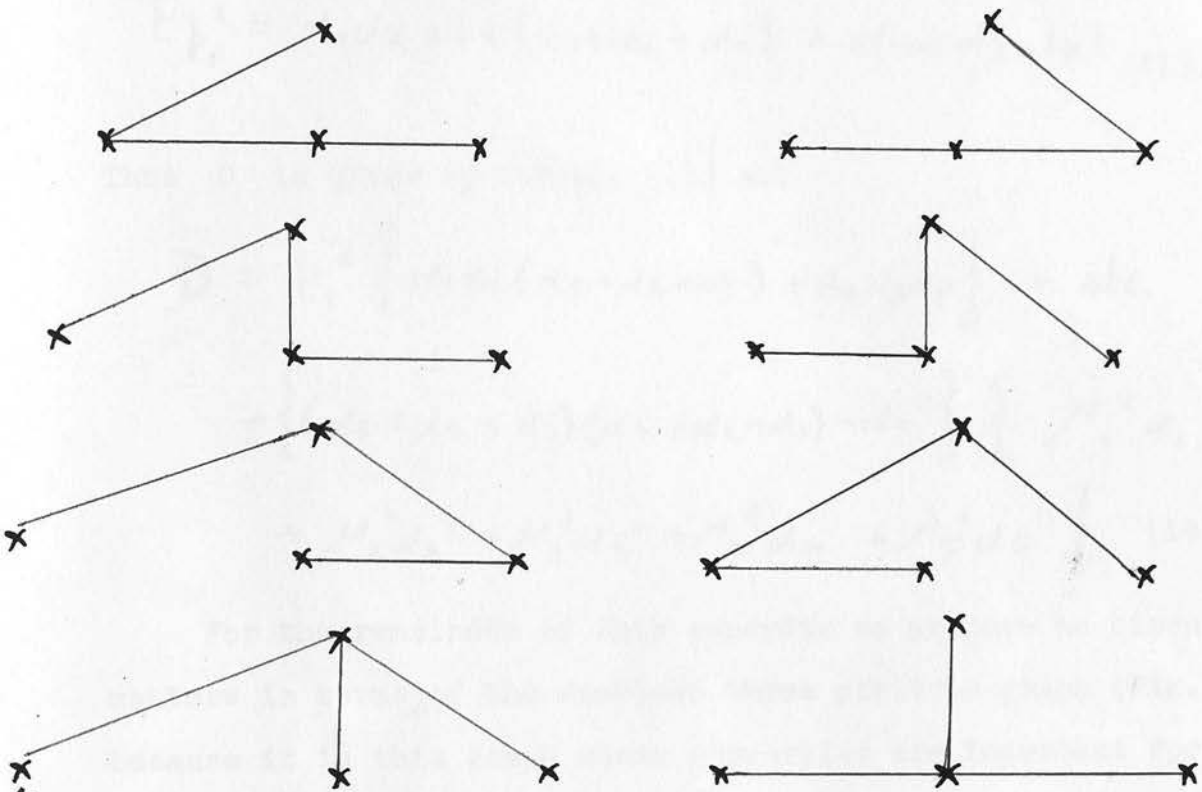


Fig. 3

products of the indicated pairs of α 's and reduces to

$$C = (\alpha_5 + \alpha_4 + \alpha_3)(\alpha_5 + \alpha_2 + \alpha_1) - \alpha_5^2 \quad (12)$$

The invariants of the graph of Fig. 2. are p_1^2 , p_2^2 , p_3^2 and the construction of $E_{p_1}^2$ is illustrated by Fig. 4.

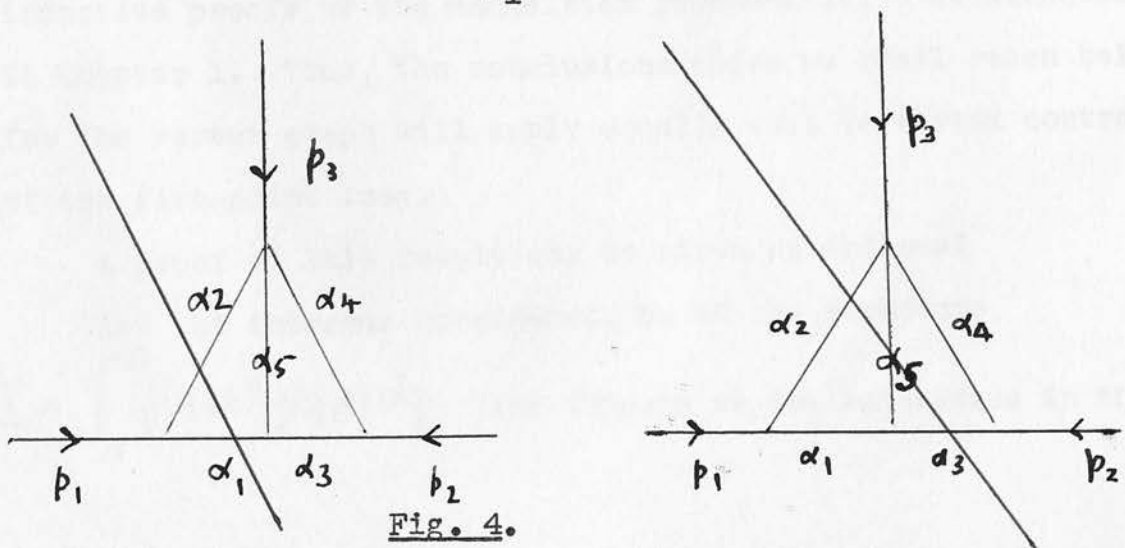


Fig. 4.

$$E_{p_1}^2 = \alpha_1 \alpha_2 \times 1 \times (\alpha_3 + \alpha_4 + \alpha_5) + \alpha_2 \alpha_5 \alpha_3 \times 1 \times 1 \quad (13)$$

Thus D is given by formula (11) as:

$$D = p_1^2 \left\{ \alpha_1 \alpha_2 (\alpha_3 + \alpha_4 + \alpha_5) + \alpha_2 \alpha_3 \alpha_5 \right\} + \text{etc.} \\ - \left\{ (\alpha_5 + \alpha_4 + \alpha_3)(\alpha_5 + \alpha_2 + \alpha_1) - \alpha_5^2 \right\} \left\{ \mu_1^2 \alpha_1 \right. \\ \left. + \mu_2^2 \alpha_2 + \mu_3^2 \alpha_3 + \mu_4^2 \alpha_4 + \mu_5^2 \alpha_5 \right\} \quad (14)$$

For the remainder of this appendix we propose to discuss matters in terms of the simplest three particle graph (Fig. 1) because it is this graph whose properties are important for the work of Chapter 2.

For any given diagram, the physical singularities of $F(z)$ which arise from various contracted diagrams are the physical singularities which one would obtain by treating each contracted graph as if it were itself a leading graph. This is to say that the physical singularities of any given graph occur as physical singularities in all higher graphs which possess this graph as a contraction. Such an assumption is implicit in the inductive proofs of the Mandelstam representation as discussed in Chapter 1. Thus, the conclusions which we shall reach below for the vertex graph will apply equally well to vertex contractions of the five-point loop.

A proof of this result can be given as follows:

Let the integral considered, be of the structure

$$I = \int_A^B f(q, z) g(q, z) dq \quad \text{and suppose we are interested in the}$$

contraction which, when treated in its own right as a leading curve has the form $I_c = \int_A^B f(q, z) dq$.

Integrate by parts:

$$I = \left[F(B, z) g(B, z) - F(A, z) g(A, z) \right] - \int_A^B F(q, z) g'(q, z) dq \quad (15)-$$

where $F(q, z) = \int^q f(q, z) dq$

Equation (15) will remain valid, as we vary z , so long as each term remains an analytic function of z .

I possesses singularities of five types

- (a) end point singularities of f
- (b) end point singularities of g
- (c) pinch singularities of f
- (d) pinch singularities of g
- (e) singularities due to both f and g .

The first term on the right hand side of equation (15) possesses branch points of the types (a) and (c) only (also poles corresponding to branch points of I of the type (b)); the second term contains all types of singularity listed above.

Now, if we wish to continue equation (15) from a region where both sides are regular to a point suspected of being a singularity of type (a) or (c), deformations of the path are necessary to avoid encountering other types of singularity.* Such deformations are irrelevant in the case of the first term on the right hand side because any two paths can be deformed

* The definition of the physical sheet prescribes the way in which the higher order branch cuts must be threaded.

into one another without crossing a singularity of the type (b), (d) or (e). Thus, in general, the left hand integral possesses the singularities of the first term of the right hand side -- and this term may be treated as if singularities other than those of I_0 did not exist.

This proves the stated result.

Even if the above theorem were false, and the physical singularities of the contracted graphs were not necessarily physical singularities for the uncontracted graph, the physical singularities of all graphs occur in their own right in the perturbation series sum (Fig. 6) and are, consequently, important in determining the analytic structure of the total amplitude.

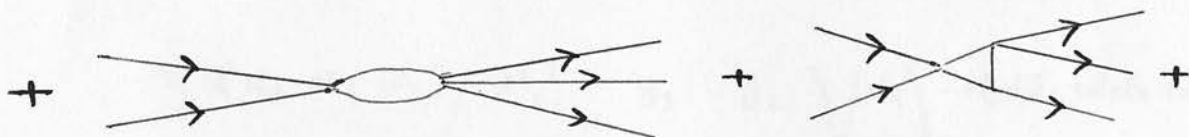


Fig. 6.

We not consider the three-point single loop graph which is described by the invariants $z_i = p_i^2$ ($i = 1, 2, 3$).

By rule, the denominator function is

$$D = z_1 d_2 d_3 + z_2 d_3 d_1 + z_3 d_1 d_2 - (\mu_1^2 - i\varepsilon_1) d_1 - (\mu_2^2 - i\varepsilon_2) d_2 - (\mu_3^2 - i\varepsilon_3) d_3 \quad (16)$$

Transforming to the variables y , via the formulae:

$$\begin{aligned} z_1 &= \mu_2^2 + \mu_3^2 - 2\mu_2\mu_3 y_1 \\ z_2 &= \mu_3^2 + \mu_1^2 - 2\mu_3\mu_1 y_2 \\ z_3 &= \mu_1^2 + \mu_2^2 - 2\mu_1\mu_2 y_3 \end{aligned} \quad (17)$$

and employing the condition

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \quad (18)$$

we obtain for the function D an expression

$$\begin{aligned} -D &= \alpha_1^2 \mu_1^2 + \alpha_2^2 \mu_2^2 + \alpha_3^2 \mu_3^2 + 2\alpha_1 \alpha_2 y_3 + 2\alpha_2 \alpha_3 y_1 \\ &\quad + 2\alpha_3 \alpha_1 y_2 - i(\alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_3 \varepsilon_3) \\ &= (\alpha_1 \alpha_2 \alpha_3) \begin{pmatrix} \mu_1^2 & y_3 & y_2 \\ y_3 & \mu_2^2 & y_1 \\ y_2 & y_1 & \mu_3^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} - i(\alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_3 \varepsilon_3) \end{aligned} \quad (19)$$

Hereafter we set $\mu_1^2 = \mu_2^2 = \mu_3^2 = 1$ for simplicity.

The various Landau curves are easily obtained. They are:

$$\begin{vmatrix} 1 & y_3 & y_2 \\ y_3 & 1 & y_1 \\ y_2 & y_1 & 1 \end{vmatrix} = 0 \quad (20)^*$$

* This curve corresponds to the second factor in equation (10).

$$\begin{vmatrix} 1 & y_i \\ y_i & 1 \end{vmatrix} = 0 \quad i=1,2,3 \quad (21)$$

The three dimensional locus (20) is a surface in the (y_1, y_2, y_3) space on which the zeros of D coincide in pairs: that subset of these points where the coincidences actually pinch the integration contour correspond to the singularities of $F(z)$. The loci (21) refer to lower order curves on which one of the α 's vanishes.

On locus (20) the values of α corresponding to the coincidence are

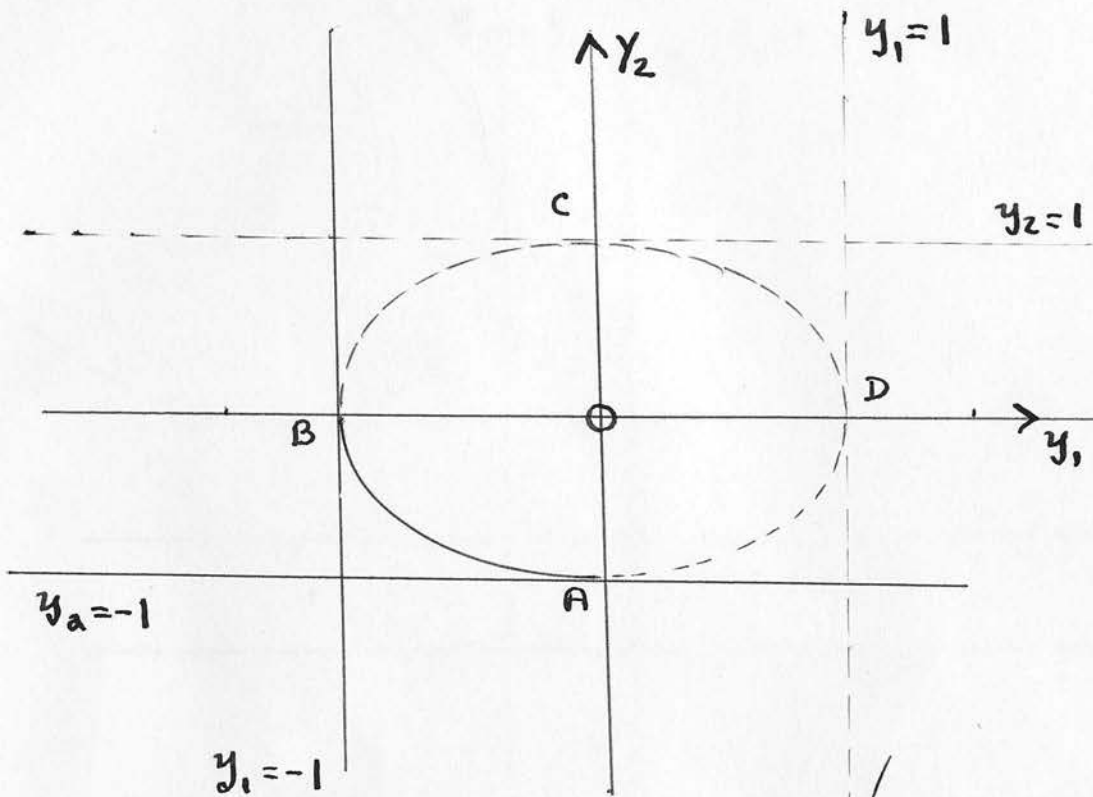
$$\frac{\alpha_1}{1 - y_1^2} = \frac{\alpha_2}{y_1 y_2 - y_3} = \frac{\alpha_3}{y_3 y_1 - y_2} \quad (22)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

On a locus of the type (21) we have, for example $\alpha_1 = 0$ corresponding to the curve $y_1^2 = 1$. If $y_1 = 1$, we have $\alpha_2 + \alpha_3 = 0$, and so, since $\alpha_1 + \alpha_2 + \alpha_3 = 1$, we must have $|\alpha_2| = |\alpha_3| = \infty$. If, on the other hand, $y_1 = -1$, we have $\alpha_2 - \alpha_3 = 0$, and so, since $\alpha_1 + \alpha_2 + \alpha_3 = 1$, we must have $\alpha_2 = \alpha_3 = \frac{1}{2}$.

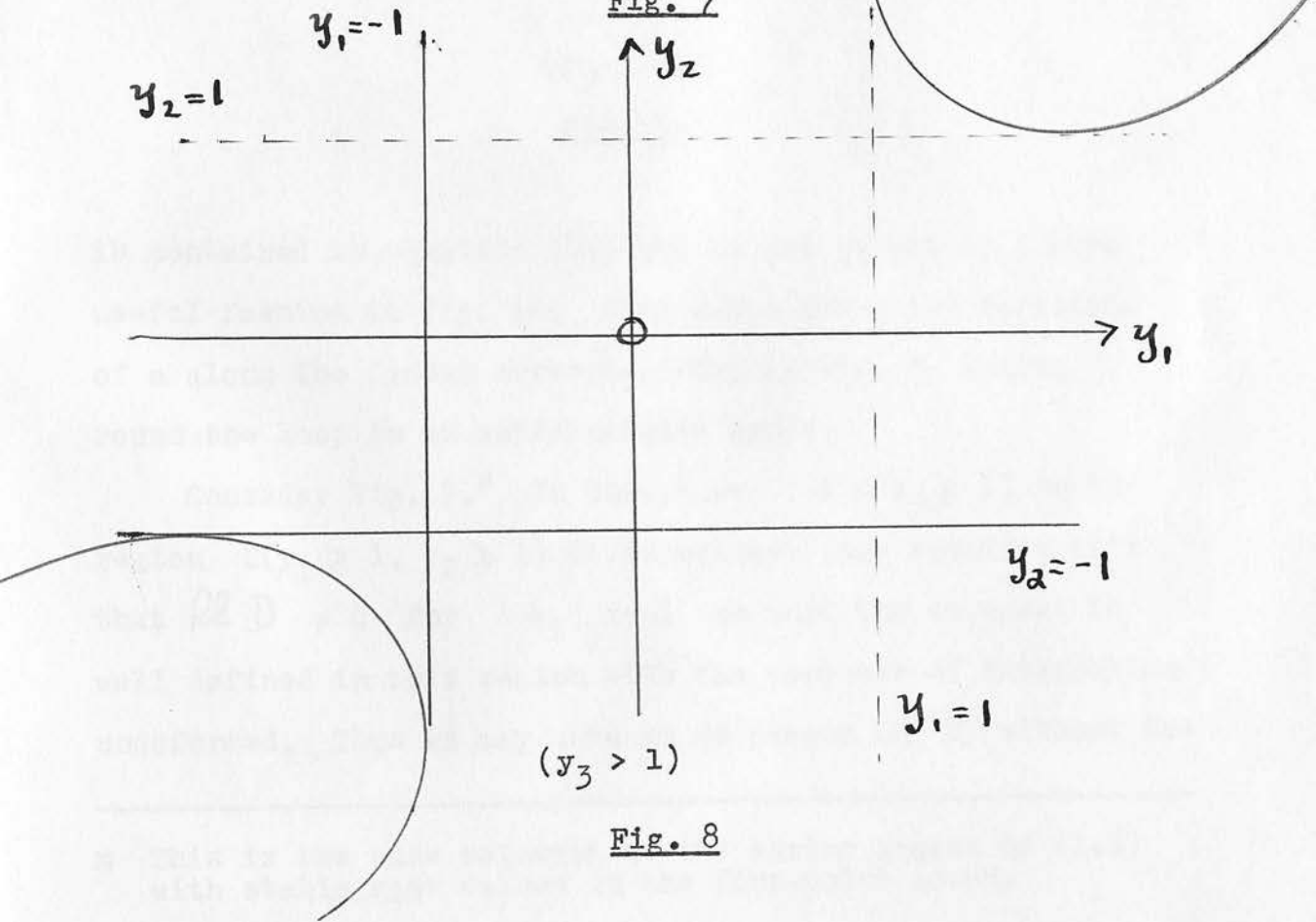
Figs. 7, 8 and 9 depict typical (y_1, y_2) sections of the leading Landau curve corresponding to $y_3 = \text{const.}$

In our discussion of analytic properties it will be necessary to know, not only the Landau curves, but also the values of the α 's at each point thereof. This information



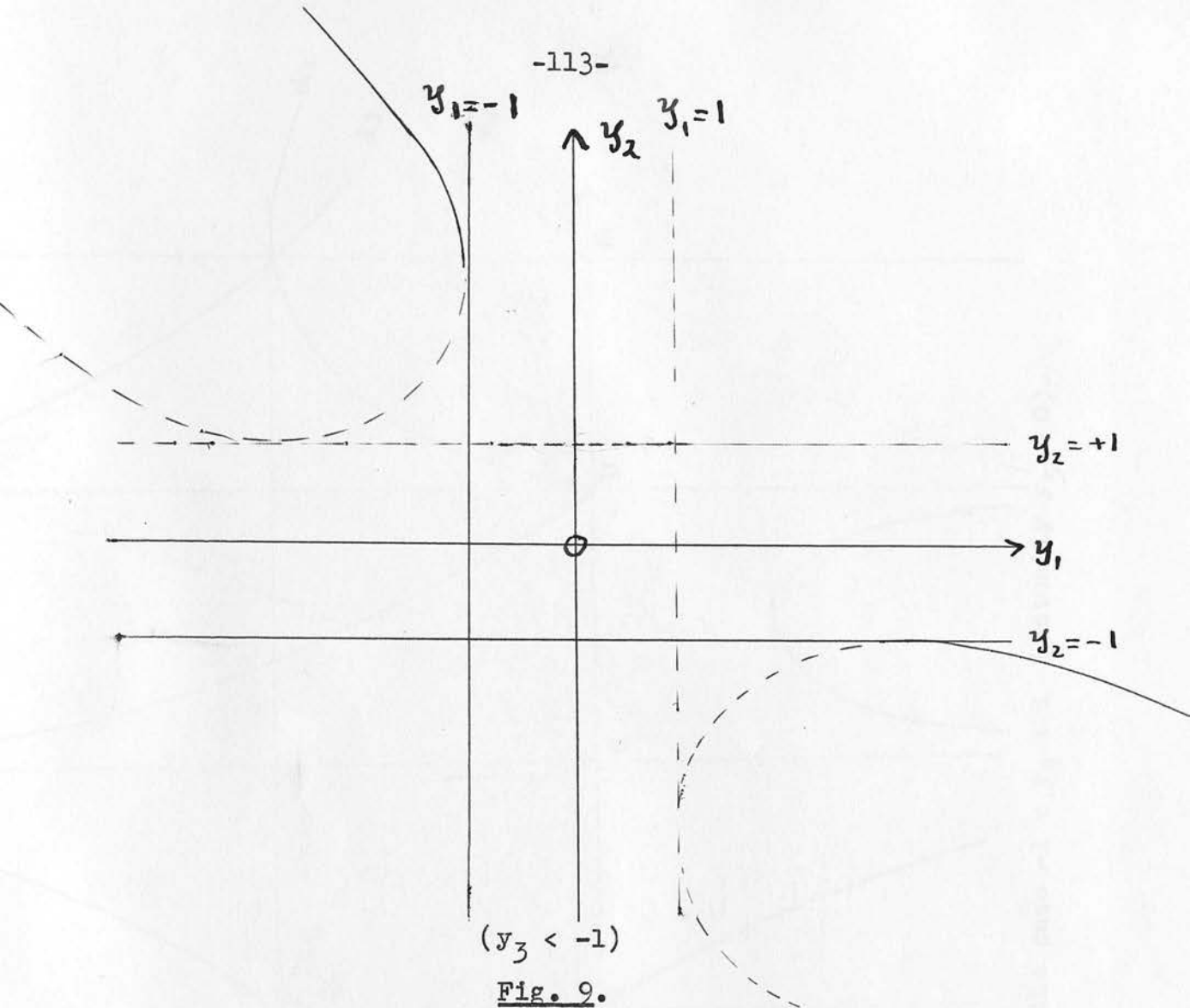
$$(-1 < y_3 < 1)$$

Fig. 7



$$(y_3 > 1)$$

Fig. 8



is contained in equation (22) but we set it out in a more useful fashion in Fig. 10. This graph shows the variation of α along the Landau curve depicted in Fig. 7, moving round the loop in an anticlockwise sense.

Consider Fig. 7.* In this case $(-1 \leq y_3 \leq 1)$ in the region $R(y_1 \geq 1, y_2 \geq 1)$ it is evident from equation (19) that $\Re D \neq 0$ for α_1 real so that the integral is well defined in this region with the contours of integration undeformed. Thus we may move as we please in R without the

* This is the case relevant to the vertex graphs of (2.6) with stable mass values in the five-point loops.

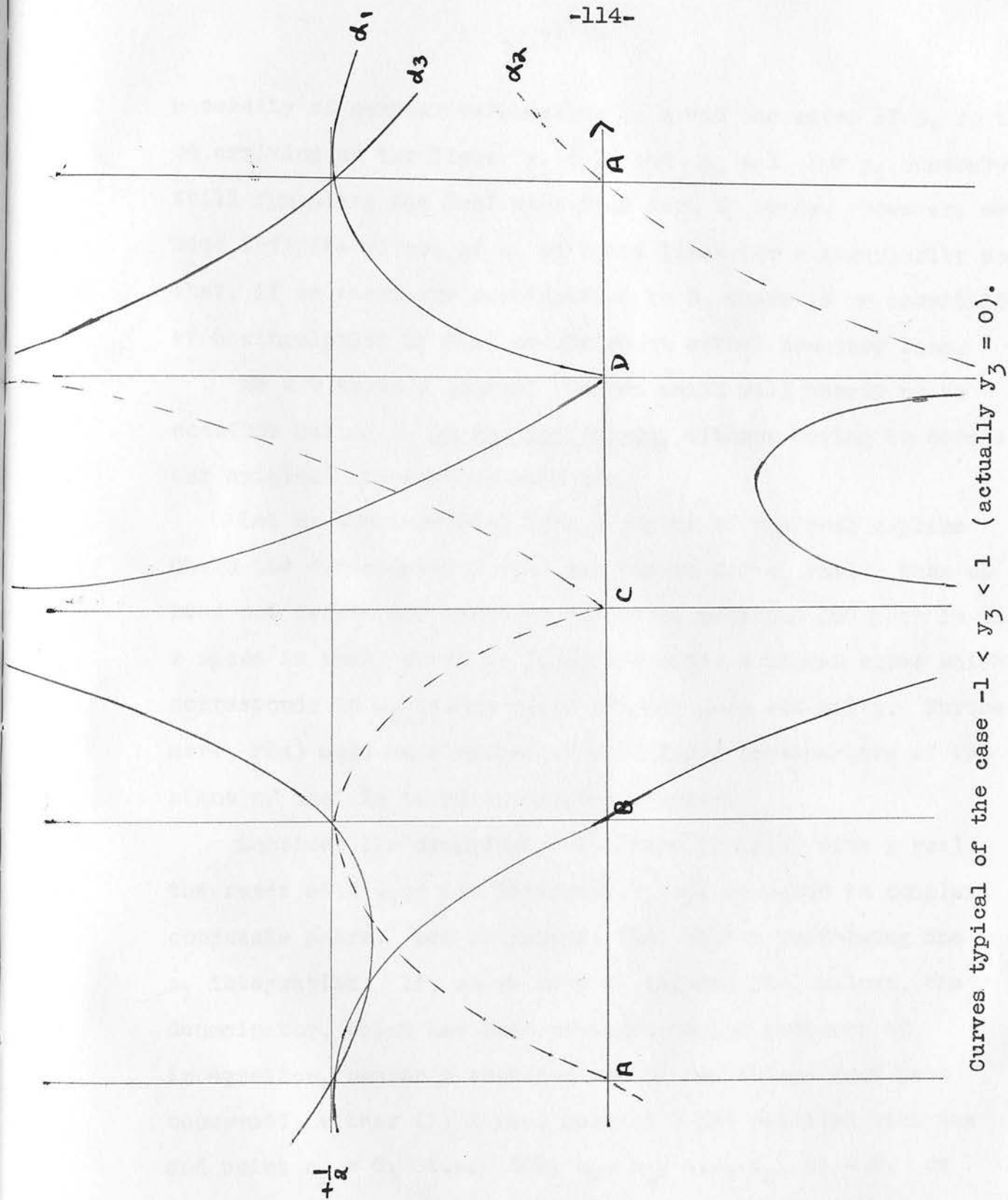


Fig. 10

necessity of contour deformation to avoid the zeros of D , so that on arriving at the lines $y_1 = 1$ and $y_2 = 1$ our α_1 contours still run along the real axis from zero to unity. However, we need infinite values of α_1 on these lines for a singularity so that, if we start our continuation in R , there is no possibility of a singularity of $F(z)$ occurring on either boundary line.

We now prove a general theorem which will permit us to continue out of R , in the real plane, without having to deform our original integration contours.

Let us continue $F(z)$ from a region of the real z -plane where the denominator D does not vanish for α_1 real: then we need not deform our original contours, provided our path in the z space is real, until we first encounter a Landau curve which corresponds to α_1 values lying between zero and unity. Furthermore, $F(z)$ must be singular at this point irrespective of the signs of the $i\epsilon$ terms in the denominator.

Consider the denominator function $D(\alpha, z)$: with z real, the zeros of $D(\alpha, z)$ are necessarily real or occur in complex conjugate pairs. Let us suppose that we are performing the α_1 integration. If, as we vary z , through real values, the denominator, which has been non-zero on our contours of integration, suddenly vanishes one of two things must have occurred: either (i) a real zero of D has collided with the end point $\alpha_1 = 0$, i.e. $D(0, \alpha_2, \alpha_3, \dots, \alpha_n, z) = 0$, or (ii) a complex conjugate pair of zeros has pinched the undeformed α_1 contour, i.e. $\frac{\partial D}{\partial \alpha_1} = 0$; in both cases (1) and

(ii) we insist that $0 \leq \alpha_i \leq 1$ because we have not deformed our contours. Now, if, when D first vanished, we had been considering the α_r integration we should have concluded that either (i) $D(\alpha_1, \dots, \alpha_{r-1}, 0, \alpha_{r+1}, \dots, \alpha_n, z) = 0$, or (ii) $\frac{\partial D}{\partial \alpha_r} = 0$ for $0 \leq \alpha_i \leq 1$. Thus, as we vary z in the real plane, when we first reach a point where the denominator vanishes in the region of integration $0 \leq \alpha_i \leq 1$, it is evident that we have encountered a Landau curve whereon $0 \leq \alpha_i \leq 1$, since a Landau curve is a locus $\alpha_i \frac{\partial D}{\partial \alpha_i} = 0$ for all $i = 1, 2, \dots, n$.

This proves the first part of the theorem.

Suppose that the above situation arises for $z = \bar{z}$ corresponding to the value $\alpha = \bar{\alpha}$ where $0 \leq \bar{\alpha}_i \leq 1$. Then, when we look at the α_1 integration either we have an end point $\alpha_1 = 0$, or a complex conjugate pair of α_1 's pinch the contour. When we proceed to the α_2 integration the critical value of α_1 is $\bar{\alpha}_1$, so that we wish to examine the α_2 integration with α_1 fixed at the real value $\bar{\alpha}_1$: in this way it is clear that either $\alpha_2 = 0$ or a complex conjugate pair of α_2 's has pinched the α_2 contour. So proceeding, we see that each integration has either an end point or pinch configuration of zeros of D : thus we conclude that $F(z)$ is singular at $z = \bar{z}$.

It is vital to realise that this result applies only to the first singularity encountered, because, to reach a second singularity, contour deformation may be required and the above argument will fail to hold. It is only the first singularity for which the α_i necessarily lie in the range

$$0 \leq \alpha_1 \leq 1.$$

Thus in the case under consideration (see Fig. 7) we may continue in the real z-plane until we reach arc CD. Consulting Fig. 10 we see that on CD the α_1 values corresponding to a coincident singularity lie outside the range $0 \leq \alpha_1 \leq 1$ and so we conclude that $F(z)$ is not singular on CD. The arguments of Section 1.2 tell us immediately that BC and DA are also regular points of $F(z)$ because the intersections at C and D are with non-singular lower order curves.

However, on reaching arc AB in the real plane, Fig. 10 now tells us that the α_1 values are in the range $0 \leq \alpha_1 \leq 1$. Thus, since this is the first time we have met such a curve, it must be a singular curve. Therefore AB is always singular irrespective of the choice of the ϵ_1 in the denominator.

In the same way we discover that the lines $y_1 = 1$, $y_2 = 1$ are singular curves. Notice that A and B are effective intersections with lower order curves and correspond to points on the leading Landau curve where the analytic properties of $F(z)$ change.

Information about singularities other than those arising for $-1 \leq y_3 \leq +1$ is quoted below.

The nature of $F(z)$ when $y_3 > 1$, in the case $\epsilon_1 > 0$, is shown in Fig. 8. The lines drawn solid are always singular while the line drawn in red is singular only in the appropriate sense (see Section (1.4)). The method used in the case $-1 \leq y_3 \leq +1$ cannot be employed in this case outside the region $y_1 \geq -1$, $y_2 \geq -1$ since these lines are singular curves. To

reach the arc drawn in red, contour deformation may be required and the signs of the ε_1 will be important.

The case $y_3 \leq -1$ is drawn in Fig. 9 and the result holds for all choices of ε_1 : the solid lines are always singular, the dotted ones always non-singular.

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